FINDING ORDER IN CHAOS, PART 2

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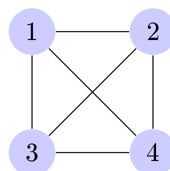
Last time we saw that K_n, the *complete graph* with n nodes — that is, a graph where every node is connected to every other — has $n(n-1)/2$ edges. The job of drawing an edge in K_n can be thought of in the following way: choose two nodes, then draw an edge between them. In other words, every edge involves two nodes, and every way of picking two nodes creates an edge. For instance, the six edges of the K_4 below correspond to the six ways of picking two of its nodes: 1 and 2; 1 and 3; 1 and 4; 2 and 3; 2 and 4; 3 and 4.



That's really where the $n(n-1)/2$ formula comes from: it simply counts the number of ways to pick 2 nodes out of n. Today we'll begin by extending our counting abilities. We'll see not only how to count groups of 2, but of 3, and 4, and any number really.

1. Imagine you have three pairs of pants, and four shirts. In how many ways can you get dressed?

2. What if you have three pairs of pants, four shirts, and two pairs of shoes?

3. You're captain of a basketball team and you need to pick your other four teammates from a pool of 20 students. You pick them by the time-honored procedure of calling out the names of the players you want, in order. In how many ways can you call out the team? (Hint: how many choices do you have for the first player? How many for the second? Etc.)

4. Imagine that, when all is said and done, your team mates are Alice, Bob, Chris, and Dana. In how many ways could you have called them out?

5*. How many different 4-player teams are there? (Hint: the difference between this problem and #3 is that here we only care *who* is on the team, and not the order they were called.)

6. You are now captain of a soccer team. Good for you, soccer is way more awesome than basketball. In any case, you need to pick 10 other team members, now from a pool of 35. The picking procedure is the same: call out their names in order. How many ways are there to call out your team? (You don't need to *do* the calculation, just indicate which calculation *should* be done.)

7. After all is said and done, your teammates are Ahmad, Bob, Carlos, Dana, Erica, Francisco, George, Haruki, Igor, and Jing. How many ways could you have called them out?

8*. How many different 10-player teams are there?

9*. How many ways are there to pick k team mates from a pool of n? (Hint: look at the pattern common to #5 and #8.)

What Are The Odds.
10. You read a news article and learn that 50% of police officers watch Glee, while 50% watch My Little Pony. Is it necessarily true that every cop watches either Glee or My Little Pony?

11. What's the smallest possible percentage of cops that watch Glee or My Little Pony? What's the largest?

12. The same news article mentions that 12% of the people in Los Angeles can ride a skateboard, 17% can ride a bike, and 15% can rollerblade. Is it possible for only 20% of Angelenos to have some skateboarding, biking or rollerblading skill? *At most* how many people can ride either a skateboard, or a bike, or rollerblade?

13. The same news article, and by now you're not sure what this article is even about, mentions that 15% of Angelenos have gotten speeding tickets at some point, 26% have gotten parking tickets, 13% have been fined for driving in a giant banana suit, 38% for mixing up "your" and "you're" while texting and driving, and 7% have had to pay a fine for letting their dog drive. These statistics probably involve a lot of overlap; the same person can get more than one kind of ticket, maybe even all of them, maybe even on the same day. For instance, you don't know how many people got speeding tickets *and no other tickets*.

Still, can you guarantee that some Angelenos have never gotten a ticket at all? If so, at least how many?

I Know I Can Do It, I Just Don't Know How.
Last time we studied *Ramsey theory*, which lets us show that even in "chaos" (an arbitrary coloring of the edges of K_n) it is possible to find small pieces that are "ordered" (monochromatic triangles and K_4s). Although we didn't go into it, Ramsey theory is a well-developed branch of modern math and lets us find much more complicated ordered things in chaotic environments. (See #9 from last time.)

Ramsey theory was the motivation for the discovery of another very strange phenomenon in mathematics. This one doesn't have a standard slogan, but here's an attempt: "if you're trying to do something complicated, sometimes you can succeed by doing it randomly".

This phenomenon was discovered when people were trying to solve the anti-Ramsey problem. That is, instead of trying to find a monochromatic triangle, they were trying to come up with a coloring *without* any monochromatic triangles. We saw that this can be done for K_5 but not for K_6; that is, there is a way to color the edges of K_5 that avoids monochromatic triangles, but the same is not true of K_6. Well, what about monochromatic K_4? We saw that any colored K_{27} has a monochromatic K_4; but can we find a coloring of K_{26} that avoids monochromatic K_4?

In the following problems, we will see the revolutionary "do it randomly" technique in action: we'll show that there is a way to color K_{11} in a way that avoids all monochromatic K_5, without actually showing what that way is. What's more, the same technique can prove that there is a way to color K_{100} in a way that avoids monochromatic K_{10}, and similarly for larger graphs. Notice that just coloring the edges of K_{100} is a chore; let alone checking that a given coloring is free of monochromatic K_{10}! So this technique lets us show that *there is* a way to do something, without getting our hands dirty with *how* to do it.

14. How many ways are there to 2-color the edges of a K_{11}?

15. How many ways are there to 2-color the edges of K_{11}, in such a way that 12345 is a monochromatic K_5? What fraction of all the colorings is that?

16. What fraction of the 2-colorings of the edges of K_{11} have 45678 a monochromatic K_5? *At most* what fraction have either 12345 *or* 45678 monochromatic (or both)? (Hint: #11. Colorings are cops, colorings where 12345 is monochromatic are cops who watch Glee, colorings where 45678 is monochromatic are cops who watch My Little Pony.)

17. How many candidates for a K_5 are there in K_{11}? That is, how many groups of 5 vertices are there? (Hint: #5, #8, and #9.)

18*. At most what fraction of colorings of K_{11} have *some* group of 5 vertices which is a monochromatic K_5? (Hint: #12. Colorings are Angelenos, and the different possibilities for a monochromatic K_5, like 12345, or 45678, or 24589, etc, are the different types of wheeled vehicle.)

19*. Show that there is *some* coloring of K_{11} where *no* group of 5 vertices is a monochromatic K_5. (That means this coloring doesn't have *any* monochromatic K_5, as promised!) (Hint: #13. Colorings are Angelenos, and the different possibilities for a monochromatic K_5, like 12345, or 45678, or 24589, etc, are the different types of ticket.)

20. Why is this method of proof referred to as "doing it randomly"? (Hint: think about picking a coloring of K_{11} totally at random. Interpret your answers to problems 10-13 as probabilities. Phrase your answer to problem 14 in the language of probability.)

21.** Can you adapt the argument we've just given to show that there is *some* coloring of K_{100} without any monochromatic K_{10}?