FINDING ORDER IN CHAOS

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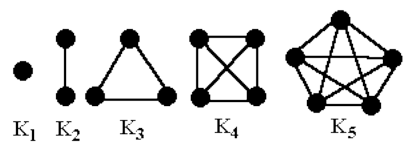
Today we are going to look at a strange phenomenon in mathematics that often goes by the slogan "order is unavoidable". It shows up in many different areas, but it was initially discovered in the context of *graph theory*. Graph theory is unique in how easy it is to explain to someone who doesn't know any math; we are all born knowing at least a little graph theory.

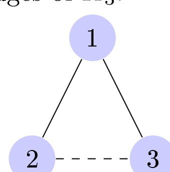
Fun and Games.

1. A *complete graph* is one that has all possible edges, that is, where any two nodes are directly connected by an edge. When a complete graph has m nodes, we call it K_m. Here are the first few complete graphs:



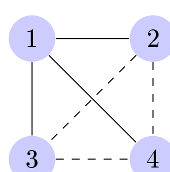
How many edges are in K_4? In K_5? Come up with a formula for the number of edges in K_m and check that it works in the first 5 cases. How many edges would be in K_{20}? What is a systematic way to draw all the edges and make sure you haven't missed any?

2. A 2-coloring of the edges of a complete graph is what it sounds like: for each edge you choose a color. Since this handout is in black-and-white, we will represent blue by a solid line, and red by a dashed line. Here is a 2-coloring of the edges of K_3:



How many differennt ways are there to 2-color the edges of K_3? Of K_5? Of K_{20}?

3. We say that a certain piece of a colored graph is *monochromatic* if all the edges in that piece are the same color. For instance, the following colored K_4 has the monochromatic triangle 234:



Can you 2-color the edges of K_4 so that there are no monochromatic triangles? How about K_5?

4. Can you 2-color the edges of K_6 so that there are no monochromatic triangles? Play a game with your classmates where you offer a 2-coloring of the edges, and they try to find a monochromatic triangle. Then switch roles.

Serious Business.

5.** You have probably noticed that the person trying to find the monochromatic triangle always wins. On the other hand, there are $2^{\binom{6}{2}} = 2^{15} = 32,768$ possible ways to 2-color the edges of K_6; you haven't tried even 1% of them all. So it's quite possible that you just haven't been clever enough to find a coloring without monochromatic triangles. Can you in fact **prove** that there is no such coloring? That is: can you prove that, *no matter* how you 2-color the edges, there will always be a monochromatic triangle *somewhere*?

6. Consider now a K_{12}. Building off the previous problem, can you prove that there are at least two monochromatic triangles? (Not necessarily of the same color.)

7*. Can you prove that a 2-coloring of the edges of K_{12} has at least two monochromatic triangles *of the same color*? (Hint: try to find as many monochromatic triangles as you can, and then later worry about finding two of the same color among them.)

8.** Now let's aim higher and try to find not just a monochromatic triangle (which is a K_3) but a monochromatic K_4. Can you show that a 2-coloring of the edges of a K_{27} necessarily contains a monochromatic K_4? (Hint: run the argument for problem 5 twice in a row.)

9*.** Show that, in any 2-coloring of the edges of the complete graph on 4^m vertices, you can find a monochromatic K_m. (So for instance you can find a monochromatic K_5 in a 2-coloring of the edges of a K_{1024}.)

10. Now that you know what Ramsey theory is about, critique the slogan "order is unavoidable". Is it a good slogan? Or can you think of a better one?

I Know I Can Do It, I Just Don't Know How.
Ramsey theory was the motivation for the discovery of another very strange phenomenon in mathematics. This one doesn't have a standard slogan, but here's an attempt: "if you're trying to do something complicated, sometimes you can succeed by doing it randomly".

This phenomenon was discovered when people were trying to solve the anti-Ramsey problem. That is, instead of trying to find a monochromatic triangle, they were trying to come up with a coloring *without* any monochromatic triangles. We saw that this can be done for K_5 but not for K_6; that is, there is a way to color the edges of K_5 that avoids monochromatic triangles, but the same is not true of K_6. Well, what about monochromatic K_4? We saw that any colored K_{27} has a monochromatic K_4; but can we find a coloring of K_{26} that avoids monochromatic K_4?

In the following problems, we will see the revolutionary "do it randomly" technique in action: we'll show that there is a way to color K_{11} in a way that avoids all monochromatic K_5, without actually showing what that way is. What's more, the same technique can prove that there is a way to color K_{100} in a way that avoids monochromatic K_{10}, and similarly for larger graphs. Notice that just coloring the edges of K_{100} is a chore; let alone checking that a given coloring is free of monochromatic K_{10}! So this technique lets us show that *there is* a way to do something, without getting our hands dirty with *how* to do it.

11. How many ways are there to 2-color the edges of a K_{11}?

12. How many ways are there to 2-color the edges of K_{11}, in such a way that 12345 is a monochromatic K_5? What fraction of all the colorings is that?

13. What fraction of the 2-coloring of the edges of K_{11} have 45678 a monochromatic K_5? What fraction have either 12345 *or* 45678 monochromatic (or both)? You don't have to find it exactly, an overestimate is enough.

14. How many candidates for a K_5 are there in K_{11}? That is, how many groups of 5 vertices are there?

15*. In the spirit of problem 11, give an overestimate for the fraction of colorings of K_{11} where *some* group of 5 vertices is a monochromatic K_5.

16*. Using problem 13, show that there is *some* coloring of K_{11} where *no* group of 5 vertices is a monochromatic K_5. (That means this coloring doesn't have *any* monochromatic K_5, as promised!)

17. Why is this method of proof referred to as "doing it randomly"? (Hint: think about picking a coloring of K_{11} totally at random. Interpret your answers to problems 10-13 as probabilities. Phrase your answer to problem 14 in the language of probability.)

18.** Can you adapt the argument we've just given to show that there is *some* coloring of K_{100} without any monochromatic K_{10}?