COUNTING WITHOUT NUMBERS

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In this meeting we'll learn about a very flexible way of counting that does not involve numbers. More precisely, we'll see how it is sometimes possible to say "in this room there are as many chairs as students" even when we don't know the number of students or the number of chairs. Extending these ideas to deal with things other than students and chairs, we'll encounter the phenomenon of *coding*: that is, of representing one thing by another, like words represent sounds, and maps represent places. The idea of coding one thing by another permeates much of mathematics, especially combinatorics and computer science.

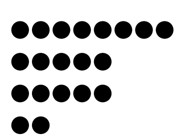
Some Simple Bijections.

1. An *incresing sequence* is an array of numbers that increase from left to right. For instance, $1, 3, 66, 88$ is an increasing sequence, whereas $1, 3, 88, 66$ is not. A *decreasing sequence* is defined similarly, except the numbers go down from left to right, as in $88, 66, 3, 1$. The *length* of a sequence is how many numbers appear in it, for instance $1, 3, 66, 88$ has length four. Imagine you're only allowed to use the numbers $1, 2, 3, \dots, 100$ to make your sequences. Can you find a bijection between increasing sequences of length seven, and decreasing sequences of length seven?

2. Consider the set $\{1, 2, 3, \dots, 100\}$ of positive whole numbers from one to a hundred. A *subset* of it is made by picking a few of the numbers that appear in it. For instance, $\{1, 7, 99\}$ and $\{2, 3, 6, 100\}$ are subsets of $\{1, 2, 3, \dots, 100\}$, as are $\{1\}$ and $\{1, 2, 3, \dots, 100\}$ itself; but $\{2, 3, 6, 101\}$ is not, since it includes 101, which is not between one and one hundred. Find a bijection that corresponds subsets of $\{1, 2, 3, \dots, 100\}$ which have 30 elements with those that have 70 elements. Can you generalize to sets other than $\{1, 2, 3, \dots, 100\}$ and sizes other than 30 and 70?

3. Consider two non-standard chessboards: one is 137 squares wide by 255 squares high, while the other is 255 squares wide by 137 squares high. Describe a simple correspondence between the squares of the first chessboard and the squares of the second. (You've proved that $137 \times 255 = 255 \times 137$ without calculating the product!)

4. A *Ferrers diagram* is an arrangement of dots in rows of non-increasing size. For example, here is a diagram with 20 dots:

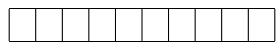


Find a bijection between Ferrers diagrams with 20 dots and 4 or fewer rows (that is, with 1, 2, 3, or 4 rows) and Ferrers diagrams with 20 dots and any number of rows, but where each row has at most 4 dots. Can you generalize?

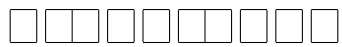
Bijections With Some Coding.

5. An *anagram* of a word is just a way to rearrange the letters in the word. For example, an anagram of EARTH is HEART; another is AEHRT. In general, words have many anagrams: THE has six and BOO has three. And the longer they are, the more anagrams they tend to have. Can you find a bijection between anagrams of SUNDAY and anagrams of PIETRO? (Note that there are too many anagrams to list by hand!)

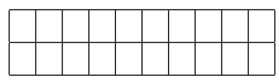
6. Imagine a rectangle which is one inch high by ten inches wide:



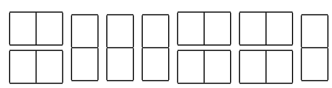
A *Fibonacci tiling* of such a rectangle is a way to cover it using only 1×1 squares and 1×2 rectangles, with no overlap. Here's an example:



Now imagine a rectangle which is *two* inches high by ten inches wide:



A *domino tiling* of it is a way to cover it with 1×2 rectangles, with no overlap. For instance:



Can you find a bijection between Fibonacci tilings of 1×10 rectangles, and domino tilings of 2×10 rectangles? Can you generalize?
Challenge. Why do Fibonacci tilings have that name? (Hint: find the number of Fibonacci tilings for a 1×1 rectangle, then a 1×2, a 1×3, ...)

7. Consider the set $\{1, 2, \ldots, 10\}$ of the positive whole numbers one through ten. It has many subsets, such as $\{1\}$, $\{3, 7, 9\}$, etc. Consider, on the other hand, 10-bit strings, that is, sequences of zeros and ones of length 10, such as 1100101101. Can you find a bijection that makes subsets of $\{1, 2, \ldots, 10\}$ correspond to 10-bit strings? How many subsets does $\{1, 2, \ldots, 10\}$ have? Generalize.

8. Everyone knows that you can only close as many brackets as you've previously opened. For instance, $(1 + (2 + 3))$ makes sense, but $1 + (2 + 3))($ is weird.

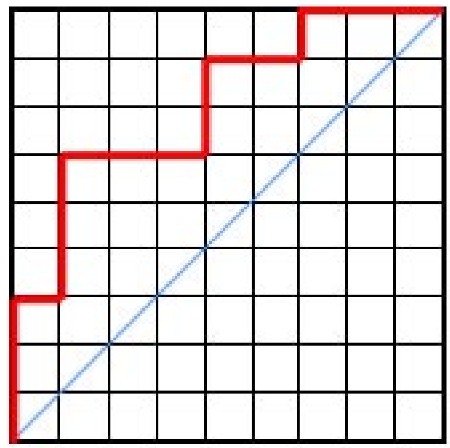
Now forget about the numbers and focus on the brackets. The pattern in the first expression is $(())$, while that in the second is $())($; we call the former *balanced*, and the latter *unbalanced*. Note that the number of open and close brackets is the same in each case! What's wrong in the second pattern is the order of the brackets.

Now imagine you have a supply of nine open brackets, and nine close brackets. There are many ways to lay them down so that they're balanced:

$$()()()()()()()()(),\ ((((((((())))))))),\ (())((())())(()()),\ \ldots$$

Call these *balanced bracketings* and keep them in mind while I tell you about a seemingly very different thing.

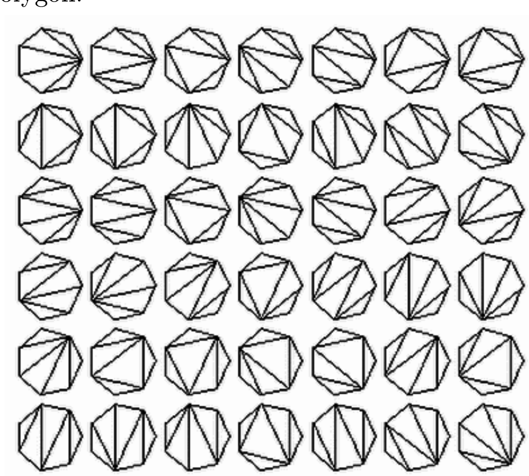
Imagine you have a 9×9 grid (see picture). A *grid walk* is a sequence of unit steps up and to the right, that starts at the lower left corner, and ends at the top right corner. For instance:



The *diagonal* of the grid is the straight line from the bottom left to the top right. We say grid walk *stays above the diagonal* if it doesn't have points strictly under the diagonal; just touching it doesn't count. The grid walk shown above stays above the diagonal.

Can you find a bijection between balanced bracketings and grid walks that stay above the diagonal? Can you generalize?

9. A *triangulation* of a regular polygon is a way to draw lines between its vertices so as to decompose it into triangles. For instance, below are all the triangulations of a seven-sided polygon:



Can you find a bijection between the triangulations of an 11-sided polygon, and balanced bracketings with 9 each of open and close brackets?