

Number Theory (Part 1)

Introduction:

#1 Which of the following integers are congruent to 4 (mod 6)?

(a) -10

(b) 20

(c) 123

#2 Determine the residue of each of the following within the given modulus.

(a) 71 (mod 3)

(b) 14 (mod 8)

(c) 3944 (mod 9)

#3 Of the 100 smallest natural numbers, how many are congruent to each of the following?

(a) 0 (mod 7)

(b) 4 (mod 7)

#4 If $a \equiv b \pmod{7}$ and $b \equiv c \pmod{7}$, then is $a \equiv c \pmod{7}$? Explain your response.

Challenge: **Prove** that if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

Addition and Subtraction:

#5 If $2 \equiv 10 \pmod{8}$
and $18 \equiv 26 \pmod{8}$
then is $20 \equiv 36 \pmod{8}$? Justify your response.

#6 If $4 \equiv 11 \pmod{7}$
and $2 \equiv 9 \pmod{7}$
then is $6 \equiv 20 \pmod{7}$? Justify your response.

#6 What general rule do you notice about modulus addition from above?

#7 Fill in the following: RULE 1: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $______ + ______ \equiv ______ + ______ \pmod{m}$
(Challenge: **Prove** that this statement is true!)

#8 Using your discovery from above, find the remainder when

(a) $6004 + 603 + 65$ is divided by 6

(b) $2 + 7 + 12 + \dots + 1002$ is divided by 5

(c) $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$ is divided by 5

(d) $17 + 177 + 1777 + 17777 + \dots + 17777777777777777777$ is divided by 8.

(e) The Fibonacci sequence is: 1, 1, 2, 3, 5, _____, _____, _____. Where the next term is the sum of the previous two terms. If we define $F_1 = 1$, $F_2 = 1$, $F_3 = 2$ and so on, find the remainder when F_{2006} is divided by 5.

Multiplication: If we can add, can we also multiply? Come up with your own numbers to try!

#9 If _____ \equiv _____ (mod 8)
 and _____ \equiv _____ (mod 8)
 then is _____ \cdot _____ \equiv _____ \cdot _____ (mod 8)?
 Justify your response.

#10 If _____ \equiv _____ (mod 5)
 and _____ \equiv _____ (mod 5)
 then is _____ \cdot _____ \equiv _____ \cdot _____ (mod 5)?
 Justify your response.

#11 RULE 2: Fill in the following: If $a \equiv b \pmod{m}$
 and $c \equiv d \pmod{m}$,
 . then _____ \cdot _____ \equiv _____ \cdot _____ (mod m) (Challenge: Explain why this statement is true!)

Exponentiation: If we can add and multiply, can we also use exponents?

#10 RULE 3: If $a \equiv b \pmod{m}$ then is $a^n \equiv b^n \pmod{m}$? Try to see if this is true for a few cases by coming up with your own numbers.
 (Challenge: Explain why this statement is true.)

#11 Using your discoveries from above, solve the following questions. **Make sure to justify your response!**

(a) Find the remainder when $17 \cdot 18$ is divided by 4.

(b) Find the remainder when $4^{18} \cdot 19^{80}$ is divided by 9

(c) Find the remainder when $100!$ is divided by 4.

(d) Is $21^{100} - 12^{100}$ a multiple of 11?

(e) The square of a positive integer leaves a remainder of 1 when divided by 5. What are the possible remainders when the integer itself is divided by 5?

(f) Find the remainder when $10^{10} + 10^{100} + 10^{1000} + \dots + 10^{10000000000}$ is divided by 7.

(g) Prove that every even square is divisibly by 4, while every odd square has remainder 1 when divided by 4.

(h) What can we say about the remainder even or odd cubes when divided by 8? What happens if we raise an even (or odd) number to the fourth power, and look at its remainder when divided by 16?

(i) More generally, suppose I start with an integer, n , and raise it to the j -th power. If I divide n^j by 2^j what remainders might I get?

Extensions:

#1 Prove that $2^{2x+1} + 1$ is divisible by 3, for any x .

#2 Prove that $4^{3x+1} + 2^{3x+1} + 1$ is divisible by 7, for any x .

#3 What are the last two digits of 3^{1234} ?

#4 Find all integers x such that $x \equiv 2 \pmod{5}$ and $x \equiv 3 \pmod{6}$?

