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## Warm-up

**Problem 1** *Many, many years ago in England, a man inherited some money and used it to start a business. In the duration of three consecutive years, he was spending £100 on the annual costs of running the business and gaining a third of what was left after taking away £100. By the end of the third year, the man doubled the original capital. What was the original capital?*

Let $x_i$ = the amount of money the man had after year $i$.

Then $x_1 = \underbrace{x_0}_{\text{initial money}} \underbrace{-100}_{\text{amount spent}} + \underbrace{\frac{x_0-100}{3}}_{\text{Amount earned.}} = \frac{4}{3}(x_0-100)$

Similarly, $x_2 = x_1 - 100 + \frac{x_1-100}{3}$, and $x_3 = x_2 - 100 + \frac{x_2-100}{3}$,

and lastly, $x_3 = 2x_0$.

The $x_2 = \frac{4}{3}(x_1-100) = \frac{4}{3}\left(\left(\frac{4}{3}(x_0-100)-100\right)\right) = \frac{16}{9}x_0 - \frac{2800}{9}$,

and $x_3 = \frac{4}{3}(x_2-100) = \frac{4}{3}\left(\frac{16}{9}x_0 - \frac{2800}{9} - 100\right)$

$= \frac{64}{27}x_0 - \frac{4\cdot 3700}{9} = \frac{64x_0 - \cancel{3700}\ 14800}{27} = 2x_0$.

$64x_0 - \cancel{3700}\ 14800 = 54x_0 \Rightarrow 10x_0 = 14800$

$\Rightarrow x_0 = 1480$

**Problem 2** *Ten children are playing in a chess tournament. Each of them plays with all others. How many games do they play in total?*

9 Player 1: Plays 9 games

8 Player 2: Plays 9, but we already counted his game against 1

7 Player 3: Plays 9, already counted games against 1 & 2.

⋮ ⋮

1

+ 0 Player 10: Already counted all of his games.

45

**Problem 3** *Which is grater, the sum of all even numbers from 0 to 10,000 or the sum of all odd numbers from 1 to 9,999? By how much? Please answer the questions without computing the sums.*

$1 + 3 + 5 + \cdots + 9{,}999$

$2 + 4 + 5 + \cdots + 10{,}000$ ⟶ Each term here

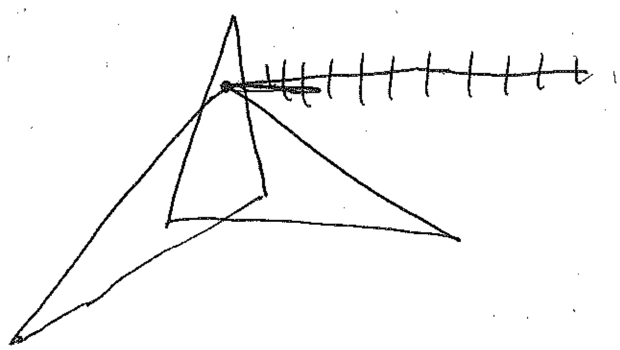
is one greater than above, and each sum has 5,000 terms, so the sum of the evens is 5,000 greater.

**Problem 4** *Draw a piecewise-linear line that has the following properties.*

*1. It is closed (the endpoint of the last segment is the beginning of the first).*

*2. It has 6 segments.*

*3. Each segment intersects only one of the remaining segments at a point different from the endpoints of the segments.*



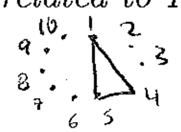
*Is a similar figure with 7 segments possible? If you think it is, please draw it. If you think it isn't, please explain why.*

Not possible. If it were, we could pair the segments up into groups of two based on which lines intersect, and since all lines intersect some line, there would be no lines left over. However, 7 is odd, so this is impossible.

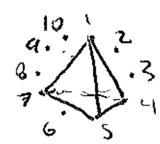
**Problem 5** *Please draw a 4D triangular pyramid.*

Note that to go from 2D → 3D, we take a 2D pyramid , add a point in a new direction , and connect all the points to it . We do the same one more time to go to 4D: 

**Problem 6** *Please draw a 9D triangular pyramid. Hint: this problem is closely related to Problem 2.*

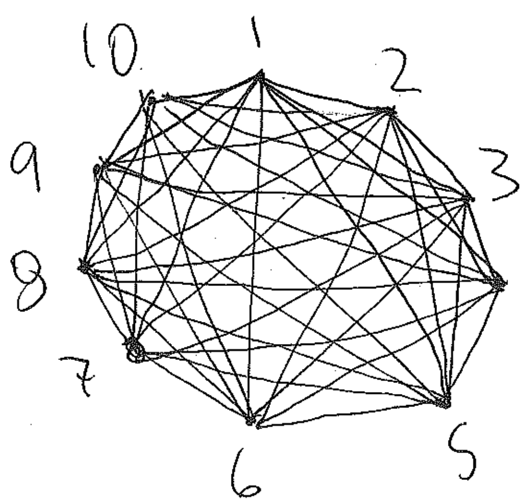
Start with points 1–10: 

Choosing 3 pts. Gives 2D pyramid.

If we choose another (and connect everything), we get a 3D cube: 

Add another to get a 4D pyramid, add another! 


In general, to get an N-dimensional pyramid, choose N+1 points on the circle and connect everything:




How many edges are there? (See problem 2).

**Time to get back to the algebra of logic...**

**Problem 7** *If possible, please expand and then simplify the expressions below according to the rules of the corresponding algebras.*

| **Algebra of Polynomials** | **Boolean Algebra** |
|---|---|
| $1+1=$ 2 | $1+1=$ 1 |
| $a+(-a)=$ 0 | $a+(\neg a)=$ 1 |
| $w\times(-w)=$ $-w^2$ | $w\times(\neg w)=$ 0 |
| $x(x+1)=$ $x^2+x$ | $x(x+1)=$ $x\cdot 1 = x$ |
| $(x+y)(x+y)=$ $x^2+2xy+y^2$ | $(x+y)(x+y)=$ $x+y$ |
| $(x+(-y))(x+(-y))=$ $x^2-2xy+y^2$ | $(x+(\neg y))(x+(\neg y))=$ $x+\neg y$ |
| $-(\underbrace{s+\ldots+s}_{56\ times})=$ $-56s$ | $\neg(\underbrace{s+\ldots+s}_{56\ times})=$ $\neg s$ |

*The problem continues on the next page.*

## Algebra of Polynomials

$$-(\underbrace{p \times \ldots \times p}_{65 \text{ times}}) = -p^{65}$$

$$a \times (a+b) = a^2 + ab$$

$$(x+y)(x+z) = x^2 + x(y+z) + yz$$

$$(x+y)(x+(-y)) =$$

$$x^2 + -xy + xy + -y^2$$

$$= x^2 - y^2$$

$$(x+y)(x+y)(x+y) =$$

$$(x^2+2xy+y^2)(x+y)$$

$$x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3$$

$$= x^3 + 3x^2y + 3xy^2 + y^3$$

## Boolean Algebra

$$\neg(\underbrace{p \times \ldots \times p}_{65 \text{ times}}) = \neg p$$

$$a \times (a+b) = a^2 + ab$$

$$= a + ab = a(1+b) = a \cdot 1 = a$$

$$(x+y)(x+z) =$$

$$\underbrace{x + xz + xy}_{} + yz$$

$$= x + yz$$

$$(x+y)(x+(\neg y)) =$$

$$x(x+\neg y) + y(x+\neg y)$$

$$= x + xy + \cancel{y \neg y} = x$$

$$(x+y)(x+y)(x+y) =$$

$$x+y$$

**Problem 8** *Which algebra, Boolean or polynomial, does the following identity belong to? Why?*

$$(x+y)(x^2 - xy + y^2) = x^3 + y^3$$

*Prove the identity in the space below.*

Polynomial. The symbol "−" doesn't make sense in Boolean algebra.

$$(x+y)(x^2 - xy + y^2) = x^3 - \cancel{x^2y} + \cancel{xy^2} + \cancel{yx^2} - \cancel{xy^2} + y^3$$

$$= x^3 + y^3$$

**Problem 9** *On the Island of Knights and Liars, knights always say the truth, liars always lie. The following conversation between Alice, Bob, Charlie, and Daniel takes place on the island.*

*Alice to Bob – You are a liar!*

*Bob to Charlie – You are a liar!*

*Charlie to Daniel – They are both liars and so are you!*

*Who are knights and who are liars?*

$A$ will mean $A$ is a knight,
$\neg A$ " $A$ is a liar, etc.

$A - \neg B$
$B - \neg C$
$C - \neg A \neg B \neg D$

Case 1. $A=1$.

Then $\neg B = 1$, so $B=0$, $C=1$, and $\neg A \neg B \neg D = 1$.

But $\neg A \neg B \neg D = 0 \neg B \neg D = 0$, so this is a contradiction.

Case 2. $A=0$.

Then $\neg B = 0$ so $B=1$, $\neg C = 1$ so $C=0$,

and thus $\neg A \neg B \neg D = 0$. Or,

$$A+B+D = \neg(\neg A \neg B \neg D) = \neg 0 = 1.$$

$A=0$, $\cancel{B=1}$ so $B+D=1$.

But, $B=1$, so $D$ could be either $0$ or $1$,

we don't know.?

So

| Knights | Liars | Unknown |
|---|---|---|
| Bob | Alice | Daniel. |
| | Charlie | |

**Problem 10** *Four boys, Peter, Quentin, Robert, and Samuel, competed in a 100 m race. The next day, they made the following statements.*

> *Peter – I was neither first nor last.*
>
> *Quentin – I was not the last.*
>
> *Robert – I was the first.*
>
> *Samuel – I was the last.*

*Three of them said the truth, one of them lied. Who lied and who won the race?*

$$P - \neg P_1 \neg P_4 \; , \; Q - \neg Q_4$$
$$R - R_1 \; , \; S - S_4$$

Case 1.

If P lied, then $\neg(\neg P_1 \neg P_4) = P_1 + P_4$ is true.
But then Robert is $1^{st}$ and Samuel is last, leaving no place for Peter. $\times$

Case 2. Q lied, so $\neg(\neg Q_4) = Q_4$ is true. But then $S_4$ is also true. $\times$

Case 3. R lied. Then $\neg R_1$ is true, so R is 2, 3, or 4. S is 4, Q is then 1, 2, or 3, and P is ~~1, 4~~ 2 or 3.
Then either 1. Q 2. R 3. P 4. S or 1. Q 2. P 3. R 4. S. Both work, so Robert lied, ~~Peter~~ Quentin was $1^{st}$.

Case 4. S lied. Then R is 1, S is 2 or 3, Q is 2 or 3, and P is 2 or 3. 3 pigeons, two holes $\rightarrow$ $\times$.

Recall that the DNF, Disjunctive Normal Form, of a Boolean algebra expression is an equivalent expression that is a sum of products of the elementary statements and their negations.

**Problem 11** *Find the DNF of the following expression.*

$$\neg(\neg AB) + \neg((A+B)(A+C) + \neg B + \neg C) =$$

$$A + \neg B + \neg(A + AC + AB + ABC + \neg B + \neg C)$$

$$A + \neg B + \neg(A(1 + C + B) + BC + \neg B + \neg C)$$

$$=$$

$$A + \neg B + \neg(A + BC + \neg B + \neg C)$$

$$=$$

$$A + \neg B + \neg A \neg(BC) B C$$

$$A + \neg B + \neg A(\neg B + \neg C) BC = A + \neg B + \neg A \cancel{\neg B} BC + \neg A \cancel{\neg C} BC$$

Recall that the FDNF, Full Disjunctive Normal Form, is the DNF such that each product has all the variables (or their negations).

**Problem 12** *Find the FDNF of the Boolean algebra expression from the previous problem.*

$$A(B + \neg B)(C + \neg C)(D + \neg D) + \neg B(A + \neg A)(C + \neg C)(D + \neg D)$$

$$A(BC + B\neg C + \neg BC + \neg B \neg C)(D + \neg D) + \neg B(AC + A\neg C + \neg AC + \neg A \neg C)(D + \neg D)$$

$$A(BCD + B\neg CD + \neg BCD + \neg B \neg CD + BC\neg D + B\neg C\neg D + \neg BC\neg D + \neg B\neg C\neg D)$$

$$+ \neg B(ACD + A\neg CD + \neg ACD + \neg A\neg CD + AC\neg D + A\neg C\neg D + \neg AC\neg D + \neg A\neg C\neg D)$$

$$=$$

$$ABCD + AB\neg CD + A\neg BCD + A\neg B\neg CD + ABC\neg D + AB\neg C\neg D$$

$$+ A\neg BCD + A\neg B\neg CD + \neg ABCD + \neg A\neg B\neg CD + A\neg BC\neg D + A\neg B\neg C\neg D + \neg A\neg BC\neg D$$

$$+ \neg A\neg B\neg C$$

Or, $AB + A\neg B + A\neg B + \neg A\neg B$

**Problem 13** *Mark the vertices of the cube below by the products from the following FDNF expression.*

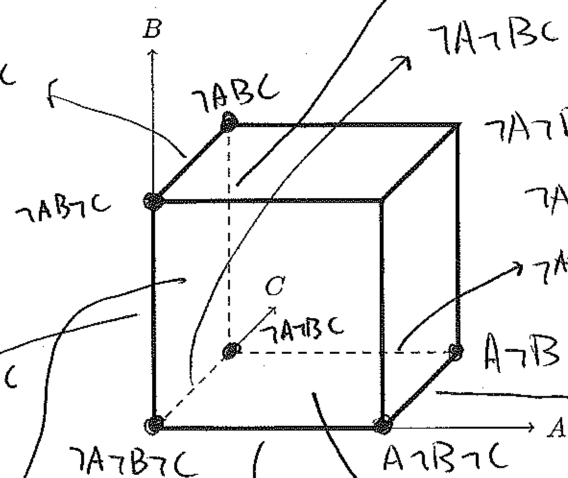
$$A\neg BC + A\neg B\neg C + \neg ABC + \neg AB\neg C + \neg A\neg BC + \neg A\neg B\neg C$$

*Use the geometric approach to simplify the sum – first simplify the sums that mark the edges, then the sums that mark the 2D faces of the 3d cube.*

$\neg ABC + \neg A\neg BC = \neg AC(B+\neg B) = \neg AC$

$\neg A\neg BC + \neg A\neg B\neg C = \neg A\neg B(C+\neg C) = \neg A\neg B$

$\neg AB\neg C + \neg ABC = \neg AB(\neg C+C) = \neg AB\cdot 1 = \neg AB$




$\neg A\neg BC + A\neg BC = (\neg A+A)\neg BC = \neg BC$

$\neg AB\neg C + \neg A\neg B\neg C = \neg A\neg C(B+\neg B) = \neg A\neg C$

$A\neg BC + A\neg B\neg C = A\neg B(C+\neg C) = A\neg B$

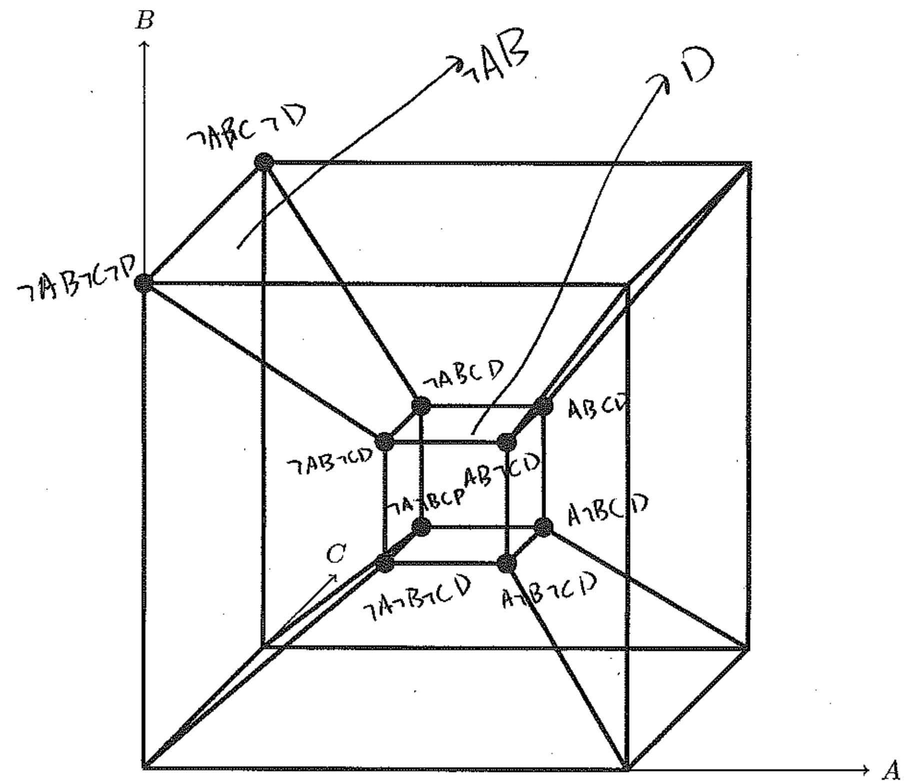
$\neg A\neg B\neg C + A\neg B\neg C = (\neg A+A)\neg B\neg C = \neg B\neg C$

$\cancel{\neg AB}C$ $\neg AB + \neg A\neg C + \neg A\neg B + \neg AC = \neg A(B+\neg B) + \neg A(\neg C+C) = \neg A + \neg A = \neg A$

$\neg B\neg C + \neg BC + A\neg B + \neg A\neg B = \neg B(\neg C+C) + (A+\neg A)\neg B = \neg B + \neg B = \neg B$

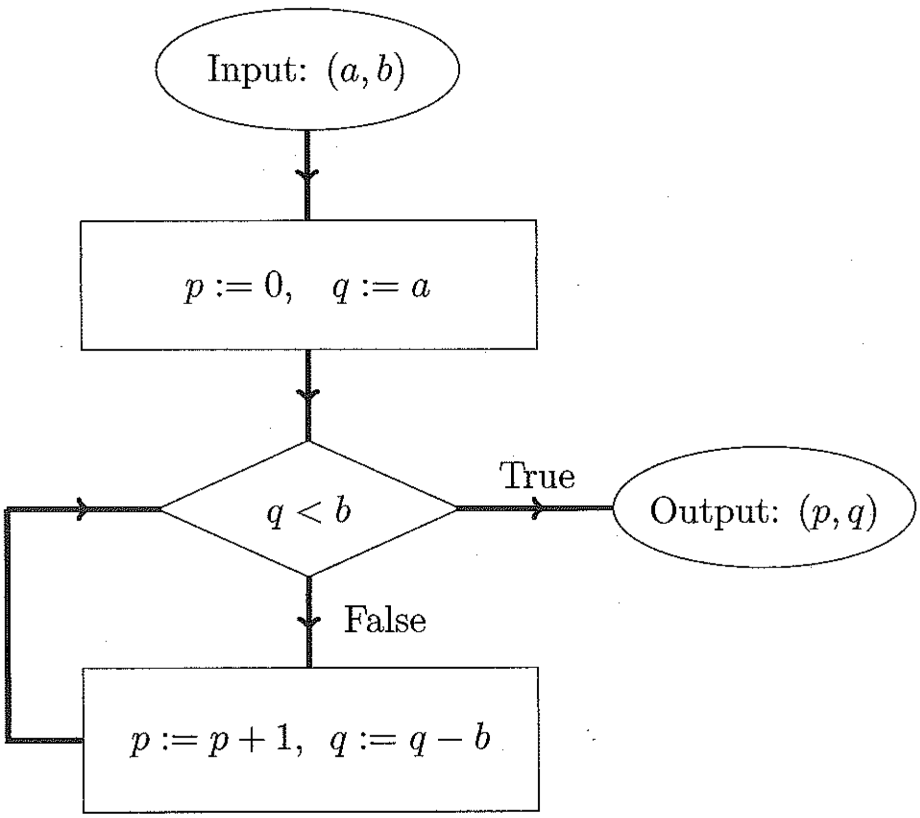
$\neg A + \neg B$

**Problem 14** *Write down the FDNF expression corresponding to the marked vertices of the following tesseract. Then use the geometric approach to simplify.*




$\neg AB + D$

Recall that the algorithm below divides a positive integer $a$ by a positive integer $b$ and finds the quotient $p$ and the remainder $q$.




**Problem 15** *Use the algorithm to divide* $1111_2$ *by* $100_2$ *without switching to the decimals.*

$p=0, q=1111$

$1111 < 100$ false.

$p=1, q = 1111 - 100 = 1011$

$1011 < 100$ false

$p=10, q = 1011 - 100 = 111$. $111 < 100$ false

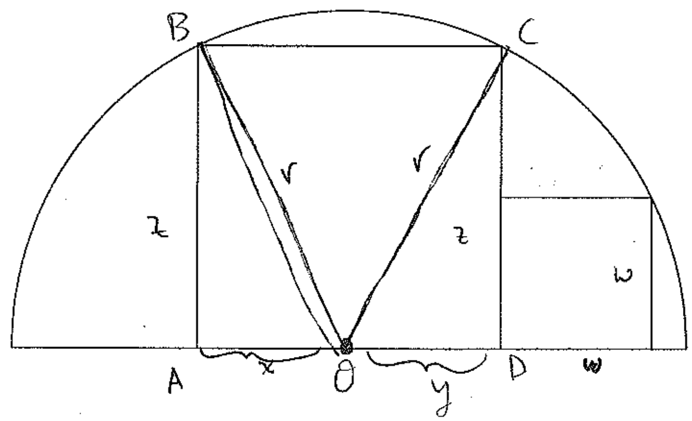
$p=11, q = 111 - 100 = 11$. $11 < 100$ true

$(11, 11)$ output.

$1111 = 11 \cdot 100 + 11$

**If you are finished doing all the above, but there still remains some time...**

**Problem 16** *Prove that the area of the larger square below is four times greater than the area of the smaller square.*




Let $O$ be the center of the circle. We wish to show that $x=y$. Draw two radii as above. Since the figure is a square, $AB=CD$ and $BAO$ and $ODC$ are right angles. Therefore, $x^2+z^2=r^2$ and $y^2+z^2=r^2$ where $z=AB=CD$ and $r$ is the radius of the semi-circle. But then $x^2+z^2=y^2+z^2$, so $x^2=y^2$, so $x=y$. Thus, the area of the big square is $(2x)^2=4x^2$. Therefore, we want to show that the area of the small square is $x^2$. However, we also know that $x^2+z^2=x^2+(2x)^2=5x^2=r^2$, so $x=\frac{r}{\sqrt{5}}$. Thus, we could also show $w=x=\frac{r}{\sqrt{5}}$.

We know that

$$(x+w)^2+w^2=r^2$$

$$\left(\frac{r}{\sqrt{5}}+w\right)^2+w^2=r^2$$

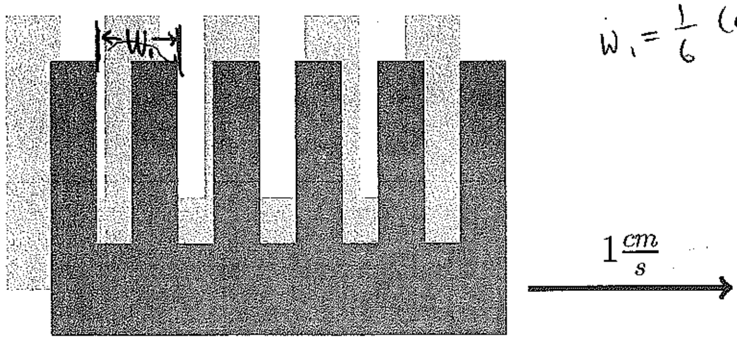
$$\frac{r^2}{5}+\frac{2r}{\sqrt{5}}w+2w^2=r^2$$

$2w^2+\frac{2r}{\sqrt{5}}w-\frac{4r^2}{5}=0$. So, by the quadratic formula:

$$w=\frac{\frac{-2r}{\sqrt{5}}\pm\sqrt{\frac{4r^2}{5}+\frac{32}{5}r^2}}{4}=\frac{\frac{-2r}{\sqrt{5}}\pm\sqrt{\frac{36r^2}{5}}}{4}=\frac{\frac{-2}{\sqrt{5}}r\pm\frac{6}{\sqrt{5}}r}{4}=\frac{\frac{4}{\sqrt{5}}r}{4}=\frac{r}{\sqrt{5}}=x.$$

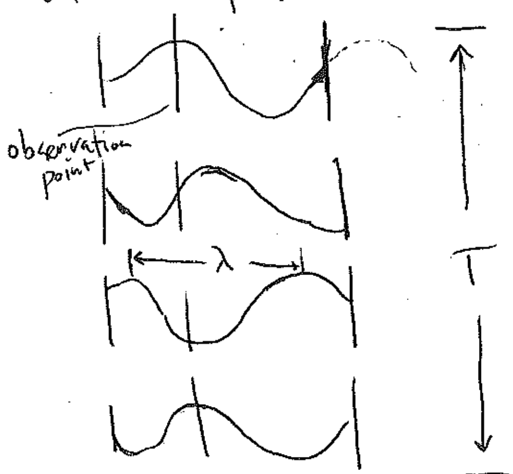
$\square$

**Problem 17** *Someone slides a comb having six teeth per centimeter against a stationary comb that has five teeth per centimeter. While sliding, the experimentator looks at a source of light through the combs. For each comb, all the teeth have equal width and so do the gaps between them. At what speed would the lit spaces (the spaces where the light gets through the combs' teeth) be moving, if the sliding comb moves at the speed of 1 centimeter per second?*




$W_1 = \frac{1}{6}$ cm

Compare to:
Calculating speed of wave on a rope.




To do this, we would figure out the time T for the pattern to repeat, find the length λ of the pattern, and compute

$v = \frac{\lambda}{T}$ to find the speed of the wave.

The pattern repeats when the comb moves $W_1$ cm. That is, in

$$T = \frac{W_1 \text{ cm}}{1 \frac{\text{cm}}{\text{s}}} = W_1 \text{ s} = \frac{1}{6} \text{ s}$$

The "length" λ of the pattern is the amount of space it takes for the pattern to repeat. This does not happen in this picture, so we need to pretend that the teeth of the comb extend beyond the 11 we can see. The repetition will happen when an integer number of teeth have passed on both combs. That is, one centimeter later. Thus, $\lambda = 1$ cm, $T = \frac{1}{6}$ s, and

$$v = \frac{\lambda}{T} = \frac{1 \text{ cm}}{\frac{1}{6} \text{ s}} = 6 \text{ cm/s}$$

Note. The wave moves faster than the comb.