

permutations with restricted positions

UCLA Math circles

November 23, 2014



objects

0	0	1	0
0	1	0	0
0	0	0	1
1	0	0	0

permutations
 $\omega = 3241$

q -analogues

0	0	0	1
0	0	1	0
1	0	0	0
0	1	0	0

permutations with
restricted positions
derangements

Outline

1. derangements

2. rook theory

3. placing rooks on Young diagrams

main reference : Enumerative Combinatorics Vol 1 2nd Ed. Sec 2.1 - 2.4

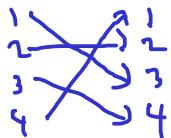
- 1 -

w permutations of $\{1, 2, \dots, n\} =: [n]$

$$w = w_1 w_2 \dots w_n \quad \text{ex. } w = 3241$$

permutation matrix of w: P_w s.t. $(P_w)_{i w(i)} = 1$
0 o/w

ex. $P_{3241} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$



- exactly one rook in each row & col.
- placement of 4 non-attacking rooks

1 → 2 → 3
↓
4 → 2 2 cycles, where there are $n(n-1)(n-2)\dots 1 = n!$ permutations of n.

permutations with restricted positions

ex. Derangements (de Montmort, Euler ~1700)
 $D_n = \#$ permutations of $[n]$ no fixed points
 $= \#$ $n \times n$ permutation matrices zero diag.

$$n=2$$

$$21$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$n=3$$

$$231, 312$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$n=4$$

$$2143, 3412, 4321$$

... six more

$$D_2 = 1$$

$$D_3 = 2$$

$$D_4 = 9$$

$D_n = \#$ permutations of $[n]$ no fixed points

let A_i = permutations where i is a fixed point

then $|A_i| = (n-1)!$

then $D_n = n! - |A_1 \cup A_2 \cup \dots \cup A_n|$

by Inclusion-Exclusion

$$\begin{aligned}
 |A_1 \cup A_2 \cup \dots \cup A_n| &= |A_1| + |A_2| + \dots + |A_n| - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots \\
 &\quad + |A_1 \cap A_2 \cap A_3| + \dots \\
 &\quad + (-1)^n |A_1 \cap A_2 \cap \dots \cap A_n| \\
 &= \sum_{k=1}^n \sum_{\substack{\text{k subsets } I \\ \text{of } \{1, 2, \dots, n\}}} \left| \bigcap_{i \in I} A_i \right|
 \end{aligned}$$

but $\left| \bigcap_{i \in I} A_i \right| = \# \text{ permutations of } n, \text{ each } i \text{ is a fixed point} = (n-k)!$

$$\begin{aligned}
 \text{So } |A_1 \cup A_2 \cup \dots \cup A_n| &= \sum_{k=1}^n (-1)^{k-1} \sum_{\substack{\text{k subsets } I \\ \text{of } \{1, 2, \dots, n\}}} (n-k)! = \sum_{k=1}^n (-1)^{k-1} (n-k)! \sum_{\substack{\text{k subsets } I \\ \text{of } \{1, 2, \dots, n\}}} 1 \\
 &= \sum_{k=1}^n (-1)^{k-1} (n-k)! \binom{n}{k}
 \end{aligned}$$

$$\begin{aligned}
 D_n &= n! - |A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{k=0}^n (-1)^k \binom{n}{k} (n-k)! \\
 &= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \right) \approx n!/e
 \end{aligned}$$

* So the probability a permutation has no fixed points is $\approx \frac{n!/e}{n!} = \boxed{\frac{1}{e}}$

- 3 -

permutations with restricted positions:
a.k.a. rook placements

Def $B \subseteq [n] \times [n]$ board

$r_k = \#$ ways of placing k non-attacking rooks
on B (rook numbers)

$N_j = \#$ permutations w of $[n]$ s.t.
 $\text{Support}(P_w) \cap B = j$ (hit numbers)

ex. derangements $B = \{(1,1), (2,2), \dots, (n,n)\}$

$D_n = N_0$

ex. $n=3$ $B = \{(1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\}$

$r_3 = 1$

$r_2 = 7$

$r_1 = 6$

$r_0 = 1$

0	1	0
1	0	0
0	0	1

0	0	0
0	1	0
0	0	1

0	1	0
0	0	0
0	0	0

0	0	0
0	0	0
0	0	0

0	P_{12}	0
P_{21}	P_{22}	P_{23}
0	P_{32}	P_{33}

Some Properties of rook placements

(A) r_k, N_j invariant under permuting rows & cols. of B

0	P_{12}	0
P_{21}	P_{22}	P_{23}
0	P_{32}	P_{33}

B

q_{11}	0	0
q_{21}	q_{22}	0
q_{31}	q_{32}	q_{33}

B'

(B) relation between r_k & N_j :

$$\text{Theorem} \quad \sum_{j=0}^n N_j x^j = \sum_{k=0}^n r_k (n-k)! (x-1)^k$$

Pf:- say $x \in \mathbb{R}$

- LHS counts placements of n rooks where rooks in B are colored with $\{1, 2, \dots, n\}$
- RHS counts placements of k rooks in B colored with $\{2, 3, \dots, n\}$ & placing $n-k$ more rooks all colored 1
- both objects are in bijection. ◻

ex. Derangements

$$B = \{(1,1), \dots, (n,n)\}$$

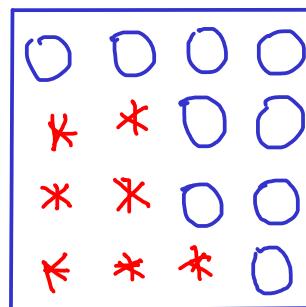
$$r_k = \binom{n}{k} \quad \text{and} \quad N_0 = \sum_{k=0}^n N_k x^k \Big|_{x=0} = \sum_{k=0}^n r_k (n-k)! (-1)^k \\ = \sum_{k=0}^n \binom{n}{k} (n-k)! (-1)^k$$

(C) Rook placements on Young diagrams

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ a partition $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq n$

$B_\lambda = \{(i,j) \mid 1 \leq i \leq n, 1 \leq j \leq \lambda_i\}$ (Young diagram)

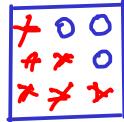
ex $n=4 \quad \lambda = (0, 2, 2, 3)$



Theorem For a Young diagram B_λ

$$\sum_{k=0}^n r_k (x)_{n-k} = \prod_{i=1}^n (x + \lambda_i - i + 1)$$

e.g. $n=3 \quad x = (1, 2, 3)$

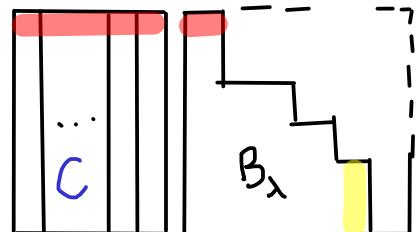


$$1(x)_3 + 6(x)_2 + 7(x)_1 + 1(x)_0 = (x+1-0)(x+2-1)(x+3-2) \\ = (x+1)^3$$

Pf. Take $x \in \mathbb{P}$

LHS counts placing n rooks on $C \cup B_\lambda$ by cols.

- placing k rooks in B_λ in r_k ways.
- placing $n-k$ rooks in C in $(x)_{n-k}$ ways



RHS counts placing n rooks on $C \cup B_\lambda$ by rows

- placing rook first row in $x + \lambda_1$ ways
- " " second " " $x + \lambda_2 - 1$ "
- ⋮

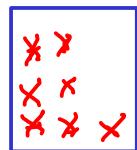


Corollary $r_n = \prod_{i=1}^n (\lambda_i - i + 1)$

(C') rook equivalence Young diagrams

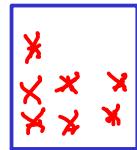
Corollary $r_k^{B_\lambda} = r_k^{B_M}$ for all k iff $\{\lambda_i - i + 1\} = \{m_i - i + 1\}$ as multisets.

ex1



$(2, 2, 3)$

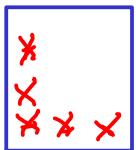
$\{2, 1, 1\}$



$(1, 3, 3)$

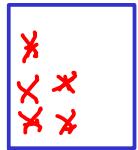
$\{1, 2, 1\}$

ex2



$(1, 1, 3)$

$\{1, 0, 1\}$



$(1, 2, 2)$

$\{1, 1, 0\}$

We say partition $\lambda \sim M$ iff $r_k^{B_\lambda} = r_k^{B_M}$

This is an interesting equivalence relation of integer partitions