

How to Fight a Dragon*

November 23th, 2014

Warm up problems

1. In a Round Robin chess tournament every two participants play exactly one game with each other. Five math circle students organized a Round Robin tournament. After each game, the winner of this game gets 2 color pencils (and the other person gets none). If a game ends in a draw, each player gets a color pencil. How many color pencils will be distributed in this tournament?

2. Put + or – sign between *some* of the digits to get a correct statement. (Note that there might be no sign between some of the digits):

$$1 \ 2 \ 3 \ 4 \ 5 \ = \ 10$$

$$1 \ 2 \ 3 \ 4 \ 5 \ = \ 38$$

$$1 \ 2 \ 3 \ 4 \ 5 \ = \ 78$$

*The idea of the problem is borrowed from the textbook “Mathematics” by L.G. Peterson (in Russian)

Far away, in a Magic Land, there are a lot of dragons and a lot of heroes. Each of the dragons has some number of Heads (H) and some number of tails (T). To defeat a dragon, a hero needs to get rid of all of its heads and all of its tails.

Here are the rules of the fight. In a single blow, a hero can cut off 1 or 2 heads, or 1 or 2 tails. A hero can not cut off a head and a tail at the same time. Moreover,

- If you cut off 1 Head, then 2 Heads immediately grow instead;
- If you cut off 2 Heads, nothing grows back;
- If you cut off 1 Tail, then 2 Tails immediately grow back;
- If you cut off 2 Tails, then 1 Head immediately grows back.

Today we will come up with a plan of defeating a dragon with a specific number of heads and tails. A dragon with H heads and T tails will be denoted by (H, T) . For example, suppose you start fighting with a dragon that has 3 heads and 5 tails. Suppose you first cut off 1 head. Then, 2 heads grow instead. As a results, the dragon now has 4 heads and 5 tails. We will write this as follows:

$$(3, 5) \xrightarrow{1H} (4, 5)$$

1. Fight the following dragons and write down what you get as a result:

- Cut 1 head from the following dragons:

- $(1, 5) \longrightarrow$;

- $(2, 4) \longrightarrow$;

- Cut 2 heads from the following dragons:

- $(2, 2) \longrightarrow$;

- $(3, 1) \longrightarrow$;

- Cut 1 tails from the following dragons:

- $(1, 5) \longrightarrow$;

- $(2, 4) \longrightarrow$;

- Cut 2 tails from the following dragons:

- $(1, 5) \longrightarrow$;

- $(2, 4) \longrightarrow$;

2. A hero is fighting with a dragon. You see the dragon in the beginning and after the first hit. Can you determine what the hero cut off in this turn? Write the answer above the arrow.

- $(2, 3) \longrightarrow (3, 3);$

- $(5, 1) \longrightarrow (5, 2);$

- $(3, 5) \longrightarrow (4, 3);$

- $(2, 4) \longrightarrow (2, 5);$

3. We can record the whole fight in this way! For example, here is a way to defeat the $(3, 0)$ dragon:

$$(3,0) \xrightarrow{1H} (4,0) \longrightarrow (2,0) \longrightarrow (0,0).$$

Describe what happens at each step in the fight (find the number of heads (H) and tails (T) that were cut off and write it above the arrow). The first step is done for you.

4. Come up with a plan to defeat the following dragons as quickly as possible:

- $(1, 0) \longrightarrow$

- $(4, 1) \longrightarrow$

- $(3, 2) \longrightarrow$

- $(1, 1) \longrightarrow$

5. The biggest dragon in the Magic Land has 2014 heads but no tails. Explain how you can defeat this dragon.

$$(2014, 0) \longrightarrow$$

6. Another huge dragon has 1 head and 2014 tails. Explain how you can defeat this dragon.

$$(1, 2014) \longrightarrow$$

7. Explain how you can defeat a dragon with an arbitrary number of heads and no tails:

- (a) First, suppose that the number of heads is even:

$$(2n, 0) \longrightarrow$$

- (b) Now suppose that the number of heads is odd. How does your strategy change?

$$(2n + 1, 0)$$

8. Explain how to defeat a dragon with 1 head and an arbitrary number of tails.

(a) First, assume the number of tails is even. (*Hint*: there are still two different cases to consider here).

(b) Now suppose that the number of tails is odd. How does your strategy change?

9. Explain how to defeat a dragon denoted by $(0, 4n)$.

10. Explain how to defeat a dragon denoted by $(1, 4n)$.

11. We will use coordinate plane to visualize the fights with the dragons:

- a dragon with H heads and T tails is be represented by a point with coordinates (H, T) on the coordinate plane.
- each move in the fight is represented by an arrow connecting the initial state of the dragon with the state of the dragon after the move.
- a dragon is defeated when it is left with 0 heads and 0 tails.

(a) For the $(3, 3)$ dragon, draw all 4 types of moves on the grid paper (next page):

$$(3, 3) \xrightarrow{1H} (4, 3)$$

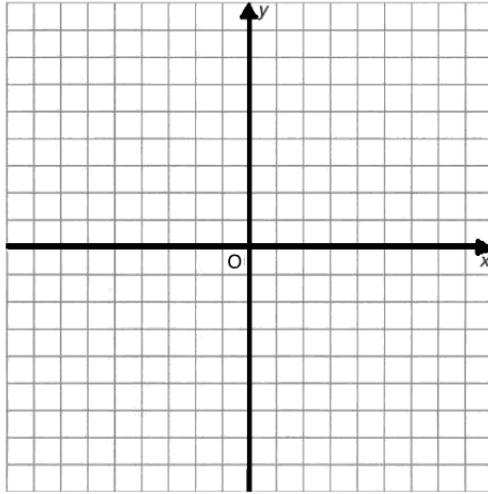
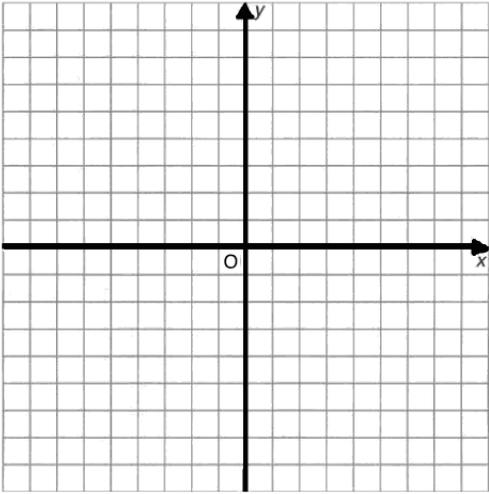
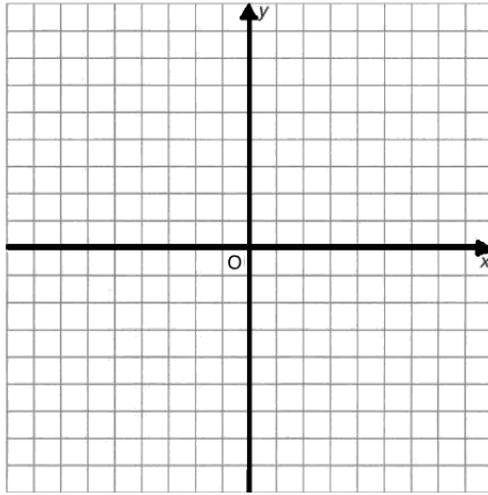
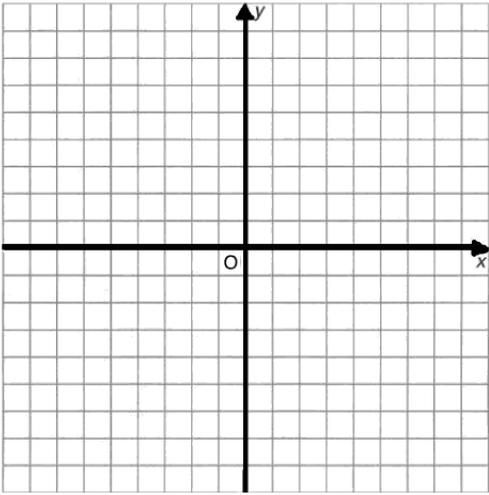
$$(3, 3) \xrightarrow{2H} (1, 3)$$

$$(3, 3) \xrightarrow{1T} (3, 4)$$

$$(3, 3) \xrightarrow{2T} (4, 1)$$

(b) On the coordinate grids below, use arrows to show how you can defeat the following dragons

- The dragon with 4 Heads and 0 Tails;
- The dragon with 1 Head and 1 Tail;
- The dragon with 3 Heads and 3 Tails;
- A dragon of your choice;



Homework

Imagine that a new hero arrives to the Magic Land. Write down detailed instructions for the hero explaining to him how defeat any Dragon in the Magic Land. Assume that the hero can instantly count the number of heads and tails of a dragon. All you need to do is to tell him what happens after each blow and (the most important part!) to explain the strategy of defeating any dragon given its number of heads and tails. Be sure to consider all possible cases and to give examples. To plan out your essay, first list all possible cases and make a plan of action in each of the cases. This is the hardest part of work. To make sure you have all possible cases, first collect data and experiment with different numbers of heads and tails and find patterns. Then generalize as much as possible.

We will carefully read your essays. The week after you give us your essays (on Dec. 7), the best essay in each group will be read to the class and will be awarded a prize. We will be looking for mathematical correctness, completeness of the solution, and quality of writing. We encourage you to typeset your story (e.g., create a Word document) and to use mathematical notation (e.g., you can use (H, T) for a dragon with H heads and T tails). Please let me know if you have any questions, and good luck with your mathematical writing!