LAMC Intermediate I & II   **November 9, 2014**

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The following is the list of the important Boolean algebra properties we have learned.

Addition is commutative.

$$A + B = B + A \tag{1}$$

Multiplication is commutative.

$$A \times B = B \times A \tag{2}$$

Addition is associative.

$$(A + B) + C = A + (B + C) \tag{3}$$

Multiplication is associative.

$$(A \times B) \times C = A \times (B \times C) \tag{4}$$

Multiplication is distributive with respect to addition.

$$A \times (B + C) = (A \times B) + (A \times C) \tag{5}$$

If the above formulas look familiar to nearly every middle school student, then most of the below are Boolean algebra specific.

Addition is distributive with respect to multiplication.

$$A + (B \times C) = (A + B) \times (A + C) \tag{6}$$

$$A + 0 = A \quad \text{and} \quad A + 1 = 1 \tag{7}$$

$$\underbrace{A + A + \ldots + A}_{\text{n times}} = A \tag{8}$$

$$A \times 0 = 0 \quad \text{and} \quad A \times 1 = A \tag{9}$$

$$\underbrace{A \times A \times \ldots \times A}_{\text{n times}} = A \tag{10}$$

$$\neg\neg A = A \tag{11}$$

The principle of the excluded middle a.k.a. the law of the excluded third: either A or $\neg A$ must be true.

$$A + \neg A = 1 \tag{12}$$

A and $\neg A$ cannot be true simultaneously.

$$A \times \neg A = 0 \tag{13}$$

Boolean algebra addition and multiplication are dual with respect to negation. The following two formulas showing it are known as De Morgan's laws.

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \tag{14}$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \tag{15}$$

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The following two identities are known as the laws of *absorption*.

$$A + AB = A \tag{16}$$

$$A \times (A + B) = A \tag{17}$$

Recall that to simplify a Boolean algebra expression means to find an equivalent expression that contains no negations of composite statements and uses as few variables as possible.

Problem 1 *Simplify the following expressions.*

- $\neg(AB + C\neg D) + \neg B + \neg C = \boxed{\neg C + \neg B + (\neg A \times D)}$

  $= \neg(AB) \times \neg(C\neg D) + \neg B + \neg C$

  $= (\neg A + \neg B) \times (\neg C + D) + \neg B + \neg C$

  $= \neg A \times \neg C + \neg A \times D + \neg B \times \neg C + \neg B \times D + \neg B + \neg C$

  $= \neg A \times \neg C + \neg C + \neg A \times D + \neg B = \neg C + \neg B + \neg B$

  $\qquad \qquad \qquad \qquad \qquad \qquad \leftarrow \neg A \times D \qquad \qquad \neg C$

- $A + A(\neg(B\neg C + \neg BD)) = \boxed{A}$

  $\underbrace{\qquad \qquad \qquad \qquad \qquad}$

  Call this P

  then $\quad A + AP = A$

- $X + XY + XYZ + X\neg YZ + XY\neg Z = \boxed{X}$

  $X + X(Y + YZ + \neg YZ + Y\neg Z)$

  $\underbrace{\qquad \qquad \qquad \qquad \qquad}$

  Call this P. Then

  $X + XP = X_3$

Problem 19 from the 11/2 handout *The year is 3014. Four kids got to the final tour of GMC8 (Galactic Math Olympiad for 8th graders), Nathan, Michelle, Laura, and Reinhardt. Some knowledgeable LAMC fans discussed their chances to win. One student thought that Nathan would take the first place and Michelle would take the second. Another student thought that Laura would take the silver while Reinhardt would end up the last of the four. The third student thought that Nathan would be second and Reinhardt third. When the results of the competition came out, it turned out that each of the LAMC students had made only one of the two predictions correct. Please find the places Nathan, Michelle, Laura, and Reinhardt got at GMC8-3014.*

Solution The following are the simple statements made above.

- $N_1 =$ *Nathan takes the first place.*

- $M_2 =$ *Michelle takes the second place.*

- $L_2 =$ *Laura takes the second place.*

- $R_4 =$ *Reinhardt takes the fourth place.*

- $N_2 =$ *Nathan takes the second place.*

- $R_3 =$ *Reinhardt takes the third place.*

Let us use the simple statements above to translate the story into the Boolean algebra language. The first fan made a composite

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statement $N_1 M_2$ that turned out to be false.

$$N_1 M_2 = 0$$

The fact that a half of the guess is true means that either $N_1 \neg M_2 = 1$ and $\neg N_1 M_2 = 0$ or that $N_1 \neg M_2 = 0$ and $\neg N_1 M_2 = 1$. This can be expressed by means of a single formula.

$$N_1 \neg M_2 + \neg N_1 M_2 = 1 \tag{18}$$

A similar translation of the other two fans' predictions into the Boolean algebra language gives us the following.

$$L_2 \neg R_4 + \neg L_2 R_4 = 1 \tag{19}$$

$$N_2 \neg R_3 + \neg N_2 R_3 = 1 \tag{20}$$

Multiplying 18, 19, and 20 brings together all the information we have about the competition.

$$(N_1 \neg M_2 + \neg N_1 M_2)(L_2 \neg R_4 + \neg L_2 R_4)(N_2 \neg R_3 + \neg N_2 R_3) = 1 \tag{21}$$

Let us first find the product of the second and third factors.

$$(L_2 \neg R_4 + \neg L_2 R_4)(N_2 \neg R_3 + \neg N_2 R_3) = 1$$

Opening parentheses gives the following.

$$L_2 \neg R_4 N_2 \neg R_3 + L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3 + \neg L_2 R_4 \neg N_2 R_3 = 1$$

Since Laura and Nathan cannot take the second place simultaneously, $L_2 \neg R_4 N_2 \neg R_3 = 0$. Since Reinhardt cannot take the third and fourth place at the same time, $\neg L_2 R_4 \neg N_2 R_3 = 0$. The above sum shortens to just two terms.

$$L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3 = 1$$

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This way, 21 boils down to the following .

$$(N_1\neg M_2 + \neg N_1 M_2)(L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3) = 1$$

Let us expand. $N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 + N_1\neg M_2\neg L_2 R_4 N_2\neg R_3 + \neg N_1 M_2 L_2\neg R_4\neg N_2 R_3 + \neg N_1 M_2\neg L_2 R_4 N_2\neg R_3 = 1$ Since $N_1 N_2 = 0$, the second term is equal to zero. Since $M_2 L_2 = 0$, the third term is equal to zero as well. Since $M_2 N_2 = 0$, the same is true for the last term. We end up with the equation
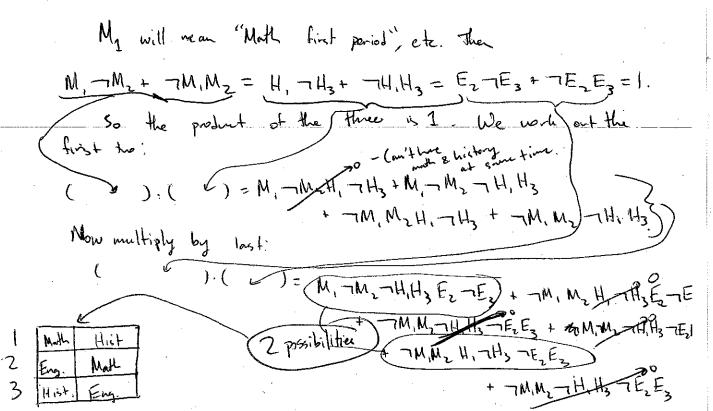
$$N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 = 1$$

that tells us the results of the competition. Nathan takes the first place, Laura the second, Reinhardt the third. Therefore, Michelle takes the fourth place. There are no contradictions: Michelle is not second, Reinhardt is not fourth, and Nathan is not second. We have solved the problem!

The following is Problem 20 from the previous handout. It is not hard to solve without using the algebra of logic. However, we will use the problem to test the power of Boolean algebra.

Problem 2 *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many different possibilities do they have?*

Please use the space at the top of the next page to work on this problem.

M_1 will mean "Math first period", etc. Then

$$\underbrace{M_1 \neg M_2 + \neg M_1 M_2}_{} = \underbrace{H_1 \neg H_3 + \neg H_1 H_3}_{} = \underbrace{E_2 \neg E_3 + \neg E_2 E_3}_{} = 1.$$

So the product of the three is 1. We work out the first two:

$(\quad) \cdot (\quad) = M_1 \neg M_2 H_1 \neg H_3 + M_1 \neg M_2 \neg H_1 H_3$
$\qquad + \neg M_1 M_2 H_1 \neg H_3 + \neg M_1 M_2 \neg H_1 H_3$

— can't have math & history at same time.

Now multiply by last:

$(\quad) \cdot (\quad) = \underbrace{M_1 \neg M_2 \neg H_1 H_3 E_2 \neg E_2}_{} + \neg M_1 M_2 H_1 \neg H_3 \neg E_2 \neg E$
$\quad + \neg M_1 M_2 \neg H_1 H_3 \neg E_2 E_3 + M_1 \neg M_2 \neg H_1 H_3 \neg E_2 1$
$\underbrace{+ \neg M_1 M_2 H_1 \neg H_3 \neg E_2 E_3}_{\text{2 possibilities}}$
$\qquad + \neg M_1 M_2 \neg H_1 H_3 \neg E_2 E_3$

Math	Hist
Eng.	Math
Hist.	Eng.

1
2
3

The LAMC old-timers have seen the following logical problems before. Now it's time to solve them again, with the help of Boolean algebra! To do so, give a name (assign a variable) to every simple statement in a problem. Then translate all the available information into some Boolean algebra formulas using the operations $+$, $\times$, and $\neg$. If possible, try to boil everything down to one formula. Simplify the formula and see if the simplified version makes the solution obvious.

Once upon a time in a land far far away, there lived a king who invented the following way of punishing criminals. Convicted lawbreakers were given a choice between two doors. Behind each door, there could be either a hungry tiger or a treasure of gold, but not nothing or both. The king would also post some warnings on the doors and then let the criminals choose.

Problem 3 *The king took the prisoner to the doors. There was a sign on each door. The first read, "There is gold in this room and there is a tiger in the other." The sign on the second door read, "There is gold in one of these rooms and in one of these rooms there is a tiger." "Are the signs true?" asked the pris-*

oner. *"One of them is,"* replied the king, *"but the other is not.*
Now, make your choice, buddy!" Which door should the prisoner
open? Why?

Let A be the statement on the first door,
B the statement on the second. Then

$$A \neg B + \neg A B = 1.$$

But note that if A is true, then B
must be as well! Therefore, $A \cdot \neg B = 0$, so

$$1 = A \neg B + \neg A B = 0 + \neg A B = \neg A B,$$

so that there is a tiger in the first
room and gold in the second.

$$\boxed{\text{Pick the second}}$$

Problem 4 *For the second prisoner, the following signs were put on the doors. Door 1: at least one of these rooms contains gold. Door 2: a tiger is in the other room. "Are the signs true?" asked the prisoner. "They are either both true or both false," replied the king. Which door should the prisoner choose? Why?*

A = statement on door 1
B = " " 2.

Then $AB + \neg A \neg B = 1$. Also, since there is either a tiger or gold behind door 2, $A + B = 1$. Therefore,

$$1 = (A + B)(AB + \neg A \neg B) = A \cdot AB + A \cdot \neg A \neg B + B \cdot AB + B \cdot \neg A \neg B$$

$$= AB + 0 + AB + 0 = AB.$$

Both signs are even true, the prisoner should choose 2

Problem 5 *In this case, the king explained that, again, the signs were either both true or both false. Sign 1: either this room contains a tiger, or there is gold in the other room. Sign 2: there is gold in the other room. Does the first room contain gold or a tiger? What about the other room?*
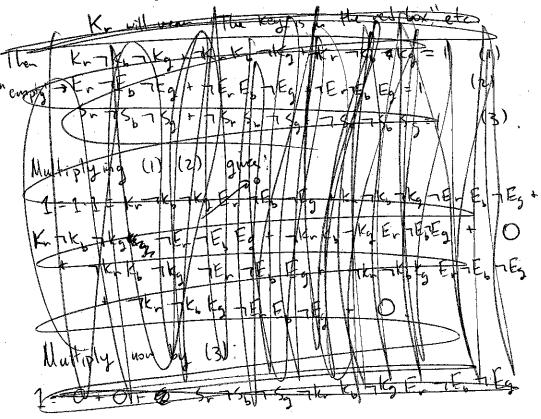
A = statement on sign 1
B = " " 2.

Again, $AB + \neg A \neg B = 1$. Again, $A + B = 1$ since there is either a tiger or gold in room 1. Doing the same multiplication as above, we get

$$1 = (A + B)(AB + \neg A \neg B) = AB,$$

so that both signs are true. This means both rooms contain gold! To see this, let G_1 mean "gold in room 1", etc. Then $A = T_1 + G_2$, $B = G_1$, so

$$1 = AB = (T_1 + G_2)G_1 = T_1 G_1 + G_2 G_1 = 0 + G_2 G_1 = G_1 G_2$$

Problem 6 *A prince travelling through a magic land found an enchanted castle guarded by an evil goblin. The goblin told the prince that he had a box with a key to the castle gate, but that it was very dangerous to open the box. The prince immediately accepted the challenge. The goblin presented the young man with three boxes, red, blue, and green. It was written on the red box, "Here is the key." The blue box read, "The green box is empty." The green box had a warning, "There is a poisonous snake in this box." "Ha-ha-ha", laughed the goblin, "it is true that one of these boxes has the key, one is home to a deadly snake and one is empty, but all the labels on the boxes lie. You can only try once. If you open the empty box, you go home empty-handed and if you open the box with the snake, you die!" Help the prince to choose wisely.*

K_R will mean "The key is in the red box," etc.

The fact that the signs lie tells us that

$K_R = E_G = S_G = 0$. (empty) Therefore, we have

$$K_B \neg K_G + \neg K_B K_G = E_R \neg E_B + \neg E_R E_B$$
$$= S_R \neg S_B + \neg S_R S_B = 1.$$

So, multiply the first two to get:

$$1 = K_B \neg K_G E_R \neg E_B + K_B \neg K_G \neg E_R E_B \quad (0)$$
$$+ \neg K_B K_G E_R \neg E_B + \neg K_B K_G \neg E_R E_B.$$

Now multiply by the last:

$$1 = K_B \neg K_G E_R \neg E_B S_R \neg S_B + K_B \neg K_G E_R \neg E_B \neg S_R S_B \quad (0)$$
$$+ \neg K_B K_G E_R \neg E_B S_R \neg S_B + \neg K_B K_G E_R \neg E_B \neg S_R S_B$$
$$+ \neg K_B K_G \neg E_R E_B S_R \neg S_B + \neg K_B K_G \neg E_R E_B \neg S_R S_B.$$

Leaves two possibilities:

Possibility #	Red	Green	Blue
1	Empty	Key	Snake
2	Snake	Key	Empty

(Pick Green!)

Problem 7 *A says, "I am a boy". B says, "I am a girl". If at least one of them is lying, who is a boy and who is a girl?*

Assume also that one is a boy and the other is a girl.

$X = A$ is a boy

$Y = B$ is a girl.

Then $\neg X + \neg Y = 1$, or better,

$$X \neg Y + Y \neg X + \neg X \neg Y = 1.$$

If $X \neg Y$ is true, then both are boys,

so $X \neg Y = 0$.

If $Y \neg X$ is true, then both are girls,

so $Y \neg X = 0$. Therefore,

$$\neg X \neg Y = 1, \text{ and}$$

A is a girl, B is a boy.

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