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Warm-up

$E=1, V=5, N=2, \text{~~~~~~~~}, S=9, R=6$ is a solution! In fact, it is the only solution, but you must continue

Recall that cryptarithmetic, also know as cryptarithm, alphametics, or word addition, is a math game of figuring out unknown numbers represented by words. Different letters correspond to different digits. Same letters correspond to same digits. The first digit of a number cannot be zero.

Case 3 (2 carry and Case 4 (3 carryover) to see why.

Problem 1 *Solve the following cryptarithm, in German.*

$$
\begin{array}{r}
E\ I\ N\ S \\
E\ I\ N\ S \\
E\ I\ N\ S \\
+\ E\ I\ N\ S \\
\hline
V\ I\ E\ R
\end{array}
$$

Check.

1	3	2	9
1	3	2	9
1	3	2	9
1	3	2	9

$S\ 3\ 1\ 6$

Works!

Solution: It is clear that E can be only 1 or 2 since otherwise the sum will be greater than 4 digits long. We split into cases based on how many digits carry over from the second column to the third.

Case 1. 0 carryover. Then we have
$0+I+I+I+I = 4I \equiv I \pmod{10}$. Ie-
$3I \equiv 0 \pmod{10}$. $3 \cdot 7 = 21 \equiv 1 \pmod{10}$, so
$7 \cdot (3I) = 2I \equiv 1I \equiv I \equiv 7 \cdot 0 = 0 \pmod{10}$,
so I must be zero. However, the only way to have an E here (since N cannot be 0=I) is if we have a carryover into the next column, which is a contradiction.

Case 2. 1 carryover. Then $1+4I \equiv I \pmod{10}$, or $3I \equiv -1 \pmod{10}$. Thus,
$I = 1 \cdot I \equiv 7 \cdot 3I \equiv -7 \equiv 3 \pmod{10}$, so $I=3$. Then
~~we have~~ $N+N+N+N+ (\text{carryover}) = 11$ or 12. The possibilities for $4N$ are ~~4,~~ 8, 12, ~~16,~~ where the crossed off ones are not possible. Thus, $N=2$ or 3.

If

Case 2a. $I=3$, $E=1$. Then we have:

$$
\begin{array}{r}
1\ 3\ N\ S \\
1\ 3\ N\ S \\
1\ 3\ N\ S \\
1\ 3\ N\ S
\end{array}
$$

We have to have $N=2$, with 3 carryover from the first column to make the 1 work. Therefore, $S=8$ or 9. But if $S=8$, then $R=S=2$, which is,

Problem 2 *Compute without a calculator.*

$$\frac{74 \times 147 - 73}{73 \times 147 + 74} = \text{Easier to see if we set } x=73. \text{ then}$$

$$\frac{(x+1) \cdot (2x+1) - x}{x \cdot (2x+1) + x+1} = \frac{2x^2 + 3x + 1 - x}{2x^2 + 2x + 1} = \frac{2x^2 + 2x + 1}{2x^2 + 2x + 1} = 1$$

Same problem!

$$\frac{244 \times 395 - 151}{244 + 395 \times 243} = \frac{244 \cdot (244 + 151) - 151}{244 + (244 + 151) \cdot 243}$$

$$= \frac{244 \cdot 244 + 243 \cdot 151}{244 \cdot 244 + 151 \cdot 243} = 1.$$

$$\frac{423134 \times 846267 - 423133}{423133 \times 846267 + 423134} =$$

$$x = 423133$$
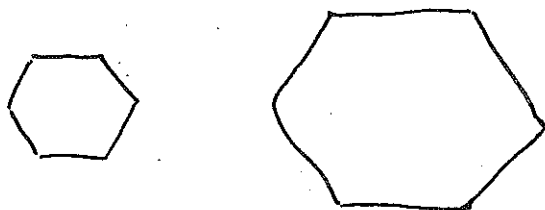
$$\frac{(x+1) \cdot (2x+1) - x}{x \cdot (2x+1) + x+1} = 1.$$

2

Recall that two figures in the (Euclidean) plane are called *similar*, if they have the same shape, but possibly different size.

Problem 3 *Draw a pair of similar hexagons in the space below.*

Problem 4 *One day, Oleg drew two similar hexagons on a paper sheet and cut them out with scissors. Oleg was quite surprised to find out that the larger hexagon never completely covered the smaller one no matter how he moved the figures on the table. Draw a pair of similar hexagons that have this property.*

Back to Boolean algebra

Problem 5 *Prove that in Boolean algebra addition is distributive with respect to multiplication.*

$$A + (B \times C) = (A + B) \times (A + C)$$

A	B	C	$B \times C$	$A + (B \times C)$	$A + B$	$A + C$	$(A + B) \times (A + C)$
0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Same!

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Negation of composite statements

The two formulas proven in Problems 6 and 8 below are fundamental for understanding the algebra of logic.

Problem 6 *Prove that* $\neg(A + B) = \neg A \times \neg B.$

A	B	$A + B$	$\neg(A + B)$	$\neg A$	$\neg B$	$\neg A \times \neg B$
0	0	0	1	1	1	1
1	0	1	0	0	1	0
0	1	1	0	1	0	0
1	1	1	0	0	0	0

Same!

Problem 7 *Negate the following statement.* **My dad likes to watch football or baseball.**

My dad does not like to watch football or baseball

OR - (Using problem 6)

My dad does not like to watch football and my dad does not like to 5 watch baseball.

Problem 8 *Prove that* $\neg(A \times B) = \neg A + \neg B$.

A	B	$A \times B$	$\neg(A \times B)$	$\neg A$	$\neg B$	$\neg A + \neg B$
0	0	0	1	1	1	1
1	0	0	1	0	1	1
0	1	0	1	1	0	1
1	1	1	0	0	0	0

Problem 9 *Negate the following statement.* **My dad likes to watch football and baseball.**

My dad does not like to watch football and baseball.

OR - (Using problem 8)

Either my dad does not like to watch football or my dad does not like to watch baseball.

The following two formulas generalize the ones proven in Problems 6 and 8.

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \quad (1)$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \quad (2)$$

Problem 10 *Simplify the following formulas so that they do not contain a negation of a composite statement.*

- $X = \neg(AB) + \neg B$

$$X = \neg A + \neg B + \neg B = \neg A + \neg B$$

- $Y = \neg(\neg BC + C)$

$$Y = \neg(\neg BC) \times \neg C = (B + \neg C) \times \neg C$$
$$= B \cdot \neg C + \neg C = (B+1)\neg C = 1 \cdot \neg C = \neg C.$$

- $Z = \neg(\neg AC) + B\neg C$

$$Z = A + \neg C + B\neg C = A + (1+B)\neg C = A + 1 \cdot \neg C = A + \neg C.$$

Problem 11 *Given the statements*

$A = $ *Bob is driving to work.*

$B = $ *Bob is shaving.*

form the statement $X = \neg(AB) + \neg B$ *from Problem 10 in plain English and simplify it in the space below.*

Either Bob is not both driving to work and shaving or Bob is not shaving.

⇓

Either Bob is not driving to work or Bob is not shaving.

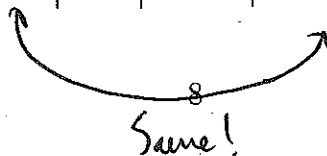## Simplifying Boolean expressions

Two expressions of Boolean algebra are called *equivalent* if they are equal as functions – the same inputs produce the same outputs. The latter can be checked by means of the corresponding truth tables. For example, the expressions $A + BC$ and $(A + B)(A + C)$ are equivalent.

Problem 12 *Prove that the expressions $A + AB$ and A are equivalent without using the truth table.*

$$A + AB = A(1+B) = A \cdot 1 = A.$$

Then use the truth table to check your proof.

A	B	AB	$A + AB$
0	0	0	0
1	0	0	1
0	1	0	0
1	1	1	1

Same!

In Problem 12, we have proven the following remarkable Boolean algebra equivalence.

$$A + AB = A \tag{3}$$

Here is one more.

Problem 13 *Prove the following formula without using the truth table.*

$$A(A + B) = A \tag{4}$$

$$A(A+B) = A \cdot A + AB = A + AB = A .$$

To *simplify* a Boolean algebra expression means to find an equivalent expression that

1. contains no negations of composite statements; and

2. has as few simple statements as possible.

The equivalence

$$A(A + \underbrace{\neg(VW + \neg XYZ)}_{B}) = A \tag{5}$$

is an example of such a simplification.

Problem 14 *If you decide to check formula 5 using a truth table, how many different inputs would you need to consider?*

A, V, W, X, Y, Z are 6 variables, each has 2 inputs. Gives 2^6 possible combinations of them.

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$\boxed{64}$

Problem 15 *Prove formula 5.*

Define a new statement B by $B = \neg(VW + \neg XYZ)$.

Then $A(A + \neg(VW + \neg XYZ)) = A(A+B) = A$ by (4).

The following two problems present two more very important equivalences.

Problem 16 *Prove the following formula.*

$$AB + A\neg B = A \tag{6}$$

$$AB + A\neg B = A(B + \neg B)$$
$$= A(1) = A.$$

Problem 17 *Prove the following formula.*

$$(A + B)(A + \neg B) = A \tag{7}$$

$$(A+B)(A+\neg B) = \underbrace{AA}_{A} + A\neg B + BA + \underbrace{B\neg B}_{0}$$
$$= A + \underbrace{AB + A\neg B}_{A} + 0$$
$$= A + A = \boxed{A}$$

Problem 18 *Simplify the following Boolean algebra expressions.*

- $\neg(AB) + \neg A =$ $\neg A + \neg B + \neg A = \boxed{\neg A + \neg B}$

- $A + \neg(\neg AB) =$ $A +$ • $A + \neg B = \boxed{A + \neg B}$

- $\neg(\neg A \neg B) + \neg A =$ $A + B + \neg A = 1 + B = B \boxed{1}$

- $\neg(A + \neg(\neg AB)) = \neg (A + A + \neg B)$

  $= \neg (A + \neg B) = \boxed{\neg A \cdot B}$

- $\neg(AB) + \neg ABC =$ $\neg A + \neg B + \neg A \cdot BC$

  $= \neg B + \neg A (1 + BC)$

  $= \neg B + \neg A (1) = \neg B + \neg A = \boxed{\neg A + \neg B}$

The problem continues on the next page.

- $\neg A + \neg(AB + \neg B) = \neg A + \neg(AB) \times B$

$$= \neg A + (\neg A + \neg B) \times B$$
$$= \neg A + \neg A \cdot B + \neg B \cdot B^0$$
$$= \neg A(1 + B) = \neg A \cdot 1 = \boxed{\neg A}$$

- $A\neg BC + A\neg(BC) + ABC + A\neg B = $

$A(\neg BC + \neg(BC))$
$"$
$A(\neg BC + \neg B + \neg C)$
$A(\neg B + \neg C)\,a$
$"$
$A \cdot \neg B + A \cdot \neg C$

$A\overline{BC + \neg B})$

$= A \cdot \neg B + A \cdot \neg C + ABC + A \cdot \neg B$
$= A(BC + \neg B + \neg C)$
$= A \cdot 1 = A$ → Check that this is 1 with a truth table!

The problem you will see below is similar to the logical problems you have solved at the Circle and at various math competitions. Please solve it any way you like. Further we will show you how to solve the problem using the Boolean algebra machinery we have developed.

Problem 19 *The year is 3014. Four kids got to the final tour of GMC8 (Galactic Math Olympiad for 8th graders), Nathan, Michelle, Laura, and Reinhardt. Some knowledgeable LAMC fans discussed their chances to win. One student thought that Nathan would take the first place and Michelle would take the second. Another student thought that Laura would take the silver while Reinhardt would end up the last of the four. The third student thought that Nathan would be second and Reinhardt third. When the results of the competition came out, it turned out that*

each of the LAMC students had made only one of the two predictions correct. Please find the places Nathan, Michelle, Laura, and Reinhardt got at GMC8-3014.

There are $4! = 24$ possibilities, so we check each:

N = Nathan   L = Laura

M = Michelle   R = Reinhardt

Need exactly 1 of the following to hold for each

(a) N=1   M=2

(b) L=2   R=4

(c) N=2   R=3

	1st	2nd	3rd	4th
Both of (a)'s predictions occur	N	M	L	R
	N	M	R	L
Both of (b)	N	L	M	R
	N	L	R	M
	N	R	M	L
	N	R	L	M
	M	N	L	R
Both of (c)	M	N	R	L
Both of (b) occur	M	L	N	R
	M	L	R	N
	M	R	N	L
	M	R	L	N
	L	N	M	R
Both of (c)	L	N	R	M
	L	M	N	R
	L	M	R	N
	L	R	N	M
	L	R	M	N
	R	N	M	L
	R	N	L	M
	R	M	N	L
	R	M	L	N
	R	L	N	M
	R	L	M	N

None of (b) occur.

None of (a) occur

So N=1, L=2, R=3, M=4

None of (b) occur

None of (b) occur

None of (c) occur

A Boolean algebra solution to Problem 19.

The following are the simple statements from Problem 19.

- $N_1 = $ *Nathan takes the first place.*

- $M_2 = $ *Michelle takes the second place.*

- $L_2 = $ *Laura takes the second place.*

- $R_4 = $ *Reinhardt takes the fourth place.*

- $N_2 = $ *Nathan takes the second place.*

- $R_3 = $ *Reinhardt takes the third place.*

Let us use the simple statements above to translate the story into the Boolean algebra language. The first fan made a composite statement $N_1 M_2$ that turned out to be false.

$$N_1 M_2 = 0$$

The fact that a half of the guess is true means that either $N_1 \neg M_2 = 1$ and $\neg N_1 M_2 = 0$ or that $N_1 \neg M_2 = 0$ and $\neg N_1 M_2 = 1$. This can be expressed by means of a single formula.

$$N_1 \neg M_2 + \neg N_1 M_2 = 1 \tag{8}$$

A similar translation of the other two fans' predictions into the Boolean algebra language gives us the following.

$$L_2 \neg R_4 + \neg L_2 R_4 = 1 \tag{9}$$

$$N_2 \neg R_3 + \neg N_2 R_3 = 1 \tag{10}$$

Multiplying 8, 9, and 10 brings together all the information we have about the competition.

$$(N_1 \neg M_2 + \neg N_1 M_2)(L_2 \neg R_4 + \neg L_2 R_4)(N_2 \neg R_3 + \neg N_2 R_3) = 1 \tag{11}$$

Let us first find the product of the second and third factors.

$$(L_2 \neg R_4 + \neg L_2 R_4)(N_2 \neg R_3 + \neg N_2 R_3) = 1$$

Opening parentheses gives the following.

$$L_2 \neg R_4 N_2 \neg R_3 + L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3 + \neg L_2 R_4 \neg N_2 R_3 = 1$$

Since Laura and Nathan cannot take the second place simultaneously, $L_2 \neg R_4 N_2 \neg R_3 = 0$. Since Reinhardt cannot take the third and fourth place at the same time, $\neg L_2 R_4 \neg N_2 R_3 = 0$. The above sum shortens to just two terms.

$$L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3 = 1$$

This way, 11 boils down to the following .

$$(N_1 \neg M_2 + \neg N_1 M_2)(L_2 \neg R_4 \neg N_2 R_3 + \neg L_2 R_4 N_2 \neg R_3) = 1$$

Let us expand. $N_1 \neg M_2 L_2 \neg R_4 \neg N_2 R_3 + N_1 \neg M_2 \neg L_2 R_4 N_2 \neg R_3 + \neg N_1 M_2 L_2 \neg R_4 \neg N_2 R_3 + \neg N_1 M_2 \neg L_2 R_4 N_2 \neg R_3 = 1$ Since $N_1 N_2 = 0$, the second term is equal to zero. Since $M_2 L_2 = 0$, the third term is equal to zero as well. Since $M_2 N_2 = 0$, the same is true for the last term. We end up with the equation

$$N_1 \neg M_2 L_2 \neg R_4 \neg N_2 R_3 = 1$$

that tells us the results of the competition. Nathan takes the first place, Laura the second, Reinhardt the third. Therefore, Michelle takes the fourth place. There are no contradictions: Michelle is not second, Reinhardt is not fourth, and Nathan is not second. We have solved the problem!

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Problem 20 *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many different possibilities do they have?*

$M_1 = $ Math class first

$M_2 = $ "   "   second ~ and so on.

We need to have

$$M_1 \neg M_2 + \neg M_1 M_2 = H_1 \neg H_3 + \neg H_1 H_3 = E_2 \neg E_3 + \neg E_2 E_3 = 1.$$

So that

$$(M_1 \neg M_2 + \neg M_1 M_2)(H_1 \neg H_3 + \neg H_1 H_3)(E_2 \neg E_3 + \neg E_2 E_3) = 1.$$

$\underbrace{M_1 \neg M_2 \cancel{H_1} \neg H_3}_{} \xrightarrow{} 0 \text{ ("Can't have M and H at same time)}$

$M_1 \neg M_2 \cancel{H_1 \neg H_3} + \underbrace{M_1 \neg M_2 \neg H_1 H_3 + \neg M_1 M_2 H_1 \neg H_3 + \neg M_1 M_2 \neg H_1 H_3}_{}$

Now multiply $\longleftarrow$ with $E_2 \neg E_3 + \neg E_2 E_3$ to get:

$M_1 \neg M_2 \neg H_1 H_3 E_2 \neg E_3 + \neg M_1 M_2 H_1 \cancel{\neg H_3} E_2 \neg E_3 \xrightarrow{} 0 \text{ (M \& E same)} + \neg M_1 M_2 \cancel{H_1 H_3 E_2 \neg E_3} \xrightarrow{} 0 \text{ (M and E same)}$

$+ M_1 \neg M_2 \neg H_1 H_3 \neg E_2 E_3 \xrightarrow{} 0 \text{ (H=E)} + \neg M_1 M_2 H_1 \neg H_3 \neg E_2 E_3 + \neg M_1 M_2 \neg H_1 H_3 \cancel{\neg E_2 E_3} \xrightarrow{} 0 \text{ (H=E)}$

(2 possibilities left)

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They are

1st	2nd	3rd
Math	English	History
History	Math	English