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Warm-up

Problem 1 What you see below is a part of the decimal multiplication table, in random order and in Japanese. Find missing factors denoted by question marks.

- (1) $\begin{array}{r} 24 \\ \times 4 \\ \hline 96 \end{array}$ $\begin{array}{r} 24 \\ \times 4 \\ \hline 96 \end{array}$ 8 (7) $\begin{array}{r} 5 \\ \times 5 \\ \hline 25 \end{array}$
(1) $futatsu \times yottsu = yattsu$ (7) $itsutsu \times itsutsu = ni\ juu\ go$
- (2) $\begin{array}{r} 8 \\ \times 9 \\ \hline 72 \end{array}$ 5 8 4 0
(2) $yattsu \times kokonotsu = nana\ juu\ ni$ (3) $itsutsu \times yattsu = yon\ juu$
- (3) $\begin{array}{r} 3 \\ \times 3 \\ \hline 9 \end{array}$ 9 2 7
(3) $mittsu \times mittsu = kokonotsu$ (9) $\begin{array}{r} 9 \\ \times 3 \\ \hline 27 \end{array}$ 2 7
(4) $\begin{array}{r} 6 \\ \times 3 \\ \hline 18 \end{array}$ 2 0 0
(4) $muttsu \times mittsu = juu\ hachi$ (10) $futatsu \times rei = rei$
- (5) $\begin{array}{r} 9 \\ \times 9 \\ \hline 81 \end{array}$ 6 9 5 4
(5) $kokonotsu \times ? = hachi\ juu\ ichi$ (11) $muttsu \times kokonotsu = ?\ go\ juu\ yottsu$.
- (6) $\begin{array}{r} 2 \\ \times 4 \\ \hline 8 \end{array}$ 1 2
(6) $yottsu \times ? = san\ juu\ ni$ (12) $futatsu \times nanatsu = ?\ juu\ yottsu$
 1 1 2
 hachi

0 1 2 3 4 5 6 7 8 9
rei ichi ni mittsu yottsu itsutsu muttsu nana yattsu kokonotsu
futatsu san go itetsu mutsu nantu hachi

Algebra of statements

A *statement* is an expression which is either *True* or *False*. For example, "Let's go!" is not a statement, while "My math teacher is not human!" is.

Problem 2 In the space below, write two sentences that are statements in the above sense and two more that are not.

Statement • Math is fun.

Statement • π is exactly three.

Not a statement • How are you?

Not a statement • Do your homework!

Warm-up

First Thought - juu appears in all numbers with more than 1 digit. It must have something to do with 10 - e.g. how many tens the number has.

In (3) we see that mittsu squared is a single digit number, but in (2) we see that $\text{mittsu} \times \text{mittsu}^2 = \text{mittsu}^3$ has more than 1 digit - ~~mittsu~~ ^{must be} ~~mittsu~~ ~~3~~.

This makes kokonotsu 9 and ni juu nana 27. We see in (2) that yattsu \times 9 = nana juu ni, which is the reverse of 27 - it must be 72. This means yattsu is 8. We ~~can~~ can make one of two guesses now about word order for 2 digit numbers -

x juu y is can be the number

$10x + y$ or $10y + x$.

We resolve this by looking at (7), which says that $(\text{itsutsu})^2 = \text{ni juu go}$. We have guessed that either $n_i = 7$ or $n_i = 2$, but there are no squares with a first digit 7, so n_i must be 2, and word order must be x juu $y = 10x + y$. This means nana is 7. The only square with 2 in the 20s is $25 = 5^2$. Therefore, it must be true that $\text{itsutsu} = 5$ and $go = 5$ (Having two names for the same digit scares us initially, but then we count how many numbers there are total - there are more than 10 given, so this might be alright).

Now we look at (4). There is nothing in front of juu, so it must be in the tens. Also, it is a multiple of mitsu = 3. Therefore, either mitsu is either 4, 5, or 6, and hachi is either 2, 5, or 8. We have two fives already, so we're comfortable ruling out that either mitsu or hachi is 5.

Now look at (1). We know yattsu is 8, so futatsu and yottsu must multiply to 8.

Next, look at (8), itsutsu \times yattsu = $8 \times 5 = 40$, so you must be 4.

Case 1. mitsu=4.

Then (4) says $4 \times 3 =$ juu hachi=12, so hachi=2. Then we can answer (5) since the only thing which multiplies with 9 to get an answer in the 20s is 3. Thus, ichi=7 and ? = 3 = mitsu. Now we have two names for 2, so nothing else can be 2. In particular, in (1), one of the factors is 8 and the other is 1. yottsu cannot be 2 (see 6). Thus, yottsu=8, futatsu=1. Then I would also have two 7s, so that in (6), san yottsu would have to be 3 (not 7). But also, we see that manatsu is almost exactly the same in spelling as nana, so these are

Case 2. Mitsu=6.

Then hachi=8, and in (5), we then see $9 \times ? = 8$. We must have ? = kokonotsu and ichi=1. In (1), mitsu \times kokonotsu = $6 \times 9 = 54$, so $\overline{?} (?) =$ go juu yon.

In (12), like I said, we think manatsu and nana are the same word but one has a suffix, thus 7. In (1), since we already have two different 8s, it seems unlikely that either of the factors is 8. One must be 4 and the other 2. In (6), we know ~~with~~ 2 digit numbers starting with one are written juu-. This, ~~the~~ san juu ni is not in the tens.

If yottsu is 2, then we would have to have san juu ni to be 12 since ni is ~~not~~ 2. Thus, yottsu=4, ?=8=hachi. San=

Then finally, in (2),

C. L. t. manatsu = $7 \times 2 = 14$ = in yot

If a statement A is true, we write $A = 1$. If a statement A is false, we write $A = 0$.

Problem 3 Determine which of the sentences below are statements and find their values.

A 23 is divisible by 5.

Statement, $A = 0$

B Please don't smoke on board the aircraft.

Not a statement

C $7x + 5y = 70$

Not a statement - cannot assign it a truth value unless we know what x, y are.

D Pyotr Tchaikovsky is a famous Russian hockey player.

Statement - ~~Not~~ $(D = 0)$

E What time is it now?

Not a statement

F Get out of here!

Not a statement.

G Math is fundamental for understanding all other sciences.

Statement, $A = 1$.

If a statement mentions only one event, true or false, it is called *simple*. If a statement mentions more than one event, it is called *composite*. For example, the statement *I come to the Math Circle by car* is simple, while the statement *I come to the Math Circle by car or by bus* is composite.

Let A and B be statements. Let us define $A + B$ as the statement A or B . For example, if $A = \text{three is greater than two}$ and $B = \text{three is greater than five}$, then $A + B = \text{three is greater than two or than five}$.

The statement A or B is false if and only if both A and B are false. If either of the statements A or B is true, then A or B is true as well.

A	B	$A + B$
0	0	0
1	0	1
0	1	1
1	1	1

We can see from the above *truth table* that in the algebra of logic $0 + 0 = 0$, while $1 + 0 = 0 + 1 = 1 + 1 = 1$.

Problem 4 Is the logical addition commutative? Why or why not? Please write down your explanation in the space below.

Yes. We only need to check that $1+0=0+1$, but both are equal to 1, so we are done.

Problem 5 Prove that $A + 0 = A$ and $A + 1 = 1$.

$$\underline{A+0=A}$$

$$\underline{\text{Case 1. } A=0.}$$

$$\text{Then } 0+0=0. \checkmark$$

$$\underline{\text{Case 2. } A=1.}$$

$$\text{Then } 1+0=1. \checkmark$$

$$\underline{A+1=1}$$

$$\underline{\text{Case 1. } A=0. \text{ Then } 0+1=1. \checkmark}$$

$$\underline{\text{Case 2. } A=1. \text{ Then } 1+1=1. \checkmark}$$

Give a verbal interpretation to the above algebraic statements.

$$A+0 = \text{or } A:$$

[Any statement] or [any ~~false~~ statement]

is ~~a true statement~~ true exactly when the first statement is.

$A+1=1$. [Any statement] or [a true statement]
is a true statement.

Problem 6 Prove that $\underbrace{A+A+\dots+A}_{n \text{ times}} = A$.

$$\underline{A=0}.$$

$$0 = 0+0 = (0+0)+0 = ((0+0)+0)+0 = \dots$$

$$\underline{A=1}.$$

$$1 = 1+1 = (1+1)+1 = ((1+1)+1)+1 = \dots$$



Problem 7 Form the logical sum of the following three statements and find its value.

$A =$ The planet of Earth rotates around the North Star.

$B =$ The planet of Earth rotates around Alpha Centauri.

$C =$ The planet of Earth rotates around the Sun.

$A + B + C =$ Either the planet Earth rotates around the North star, the planet Earth rotates around Alpha Centauri, or the planet Earth rotates around the sun.

$$A=B=0, C=1.$$

$$(A+B)+C = (0+0)+1 = 0+1 = 1$$

Problem 8 Prove that for the logical addition, $(A + B) + C = A + (B + C)$. Hint: use the truth table below.

A	B	C	$A + B$	$B + C$	$(A + B) + C$	$A + (B + C)$
0	0	0	0	0	0	0
1	0	0	1	0	1	1
0	1	0	1	1	1	1
0	0	1	0	1	1	1
1	1	0	1	1	1	1
1	0	1	1	1	1	1
0	1	1	1	1	1	1
1	1	1	1	1	1	1



These columns are the same,
 $\therefore (A+B)+C = A+(B+C)$.



Similar to the logical addition, we can introduce logical multiplication. Let us define $A \times B$ as the statement A and B . For example, if $A = \text{three is greater than two}$ and $B = \text{three is greater than five}$, then $A \times B = \text{three is greater than two and five}$. Quite obviously, A and B is true if and only if both A and B are true.

A	B	$A \times B$
0	0	0
1	0	0
0	1	0
1	1	1

In the algebra of logic just like in the algebra of numbers, $0 \times 0 = 1 \times 0 = 0 \times 1 = 0$, while $1 \times 1 = 1$.

Problem 9 Is the logical multiplication commutative? Why or why not? Please write down your explanation in the space below.

Yes, again we only need to check
that $1 \times 0 = 0 \times 1$ (since anything commutes with itself),
but both are equal to 0 so we're done.

Problem 10 Prove that $A \times 0 = 0$ and $A \times 1 = A$.

A	0	$A \times 0$
0	0	0
1	0	0

Same

A	1	$A \times 1$
0	1	0
1	1	1

Same

Problem 11 Prove that $\underbrace{A \times A \times \dots \times A}_{n \text{ times}} = A$

$$A=0.$$

$$0=0 \times 0 = (0 \times 0) \times 0 = ((0 \times 0) \times 0) \times 0 = \dots$$

$$A=1.$$

$$1=1 \times 1 = (1 \times 1) \times 1 = (((1 \times 1) \times 1) \times 1) = \dots$$

Problem 12 Prove that for the logical multiplication,
 $(A \times B) \times C = A \times (B \times C)$.

A	B	C	$A \times B$	$B \times C$	$(A \times B) \times C$	$A \times (B \times C)$
0	0	0	0	0	0	0
1	0	0	0	0	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
1	1	0	1	0	0	0
1	0	1	0	0	0	0
0	1	1	0	1	0	0
1	1	1	1	1	1	1

Same.

Problem 13 Form the logical product of the following three statements and find its value.

$A = \text{Lobsters live in the ocean.}$

$B = \text{Mobs live in the ocean.}$

$C = \text{Lobsters are no better than mobsters.}$

$A \times B \times C = \text{Lobsters live in the ocean, mobsters do not live in the ocean, and lobsters are no better than mobsters.}$

$$A=1, B=0, C=0.$$

$$(1 \times 0) \times 0 = 1 \times 0 = 0.$$

Problem 14

$A = I$ start to like the science of logic.

$B =$ "To Kill a Mockingbird" is a hunters' guidebook.

$C = 48$ is divisible by 12.

Form the following statements from the above A , B , and C and find their value.

$A \times (B + C) =$ I'm starting to like the science of logic and either "To Kill a Mockingbird" is a hunter's guidebook or 48 is divisible by 12.

$$A=1, B=0, C=1$$

$$A \times (B+C) = 1 \times (0+1) = 1 \times 1 = 1$$

$A \times B + A \times C =$ Either I'm starting to like the science of logic and "To Kill a Mockingbird" is a hunter's guidebook, or I'm starting to like the science of logic and 48 is divisible by 12.

$$A \times B + A \times C = 1 \times 0 + 1 \times 1 = 0 + 1 = 1$$

Problem 15 Prove that

$$A \times (B + C) = (B + C) \times A = A \times B + A \times C.$$

This is trivial since we have already proved commutativity, so we show only $A \times (B+C) = A \times B + A \times C$.

A	B	C	$B + C$	$A \times B$	$A \times C$	$A \times (B+C)$	$A \times B + A \times C$
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
0	1	1	1	1	0	0	0
1	1	1	1	1	1	1	1

Same!

A statement A preceded by *it is not true that ...* or a statement equivalent to such is called the *negation* of A and is denoted as $\neg A$. For example, the negation of the statement B from Problem 14 reads as follows. "*To Kill a Mockingbird*" is not a hunters' guidebook. The following is the truth table for the negation.

A	$\neg A$
0	1
1	0

Problem 16 Write down the negation of the statement "I come to the Math Circle by car or by bus" in the space below.

I do not come to Math Circle by either
car or bus.

Or

I come to Math Circle by neither car
nor bus.

Problem 17 Write down your own composite statement and its negation.

Statement: I am a math major and I like the color blue.

Negation: Either I am not a math major or
I do not like the color blue.

Problem 18 Can the statement $A + \neg A$ be false? Why or why not?

No! One of the two statements A and $\neg A$ must be true, and anything or a true statement is true.

A	$\neg A$	$A + \neg A$
0	1	1
1	0	1

Write the fact down as an algebraic formula.

$$A + \neg A = 1$$

Problem 19 Can the statement $A \times \neg A$ be true? Why or why not? No. One of the two must be false, and anything and a false statement is false.

A	$\neg A$	$A \times \neg A$
1	0	0
0	1	0

Write the fact down as an algebraic formula.

$$A \times \neg A = 0$$

Problem 20 Find the following.

$$\neg \neg A = A$$

A	$\neg A$	$\neg(\neg A)$
0	1	0
1	0	1

Same

Problem 21 Given the statements

A: Bob is driving to work.

B: Bob is shaving.

C: Bob is eating a burger.

form the following statements.

- $AB + \neg C =$ Either Bob is driving to work and shaving
or Bob is not eating a burger.
- $(A + B)C =$ Bob is either driving to work
or shaving and Bob is eating a burger.
- $A\neg B + C =$ Either Bob is driving to work and not
shaving or Bob is eating a burger.
- $\neg A\neg BC =$ Bob is not driving to work, not shaving,
and eating a burger.

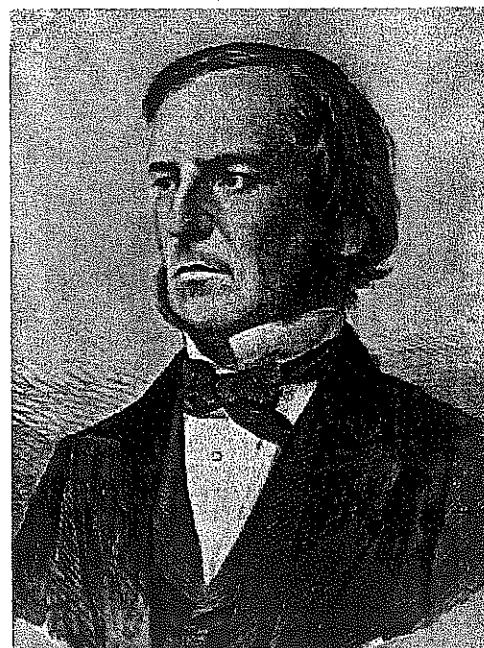


Problem 22 Using the simple statements A , B , and C from Problem 21, rewrite the following as a mathematical formula.

It is not true that Bob is either driving to work and shaving or eating a burger.

$$\neg ((A \times B) + C)$$

The algebra of logic we have studied above is called *Boolean*, after George Boole (1815-1864), an English mathematician, philosopher and logician.



George Boole

Below you will find one more feature of the algebra that truly distinguishes it from every other algebraic structure you have seen before.

In Problem 15, we have proven that, similar to the algebra of numbers, multiplication in Boolean algebra is distributive.

$$A \times (B + C) = A \times B + A \times C$$

In Boolean algebra, unlike the algebra of numbers, addition is distributive with respect to multiplication as well!

$$A + (B \times C) = (A + B) \times (A + C) \quad (1)$$

Problem 23 Prove formula 1.

A	B	C	$B \times C$	$A+B$	$A+C$	$A+(B \times C)$	$(A+B) \times (A+C)$
0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	1
0	1	0	0	1	0	0	0
0	0	1	0	0	1	0	0
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Same!

Problem 24 Is the sentence "this statement is false" a statement? If you think it is, find its value. If you think it's not, give a reason.

Not a statement!

If the statement is true, then it is also false.

If it is false, then it is also true.

However, we must be able to assign exactly 1 value 1 or 0 to anything we consider a statement, but here we must assign both 1 and 0.

For more, see Russell's Paradox on Wikipedia.

If you are finished doing all the above, but there still remains some time ...

... let us get back to

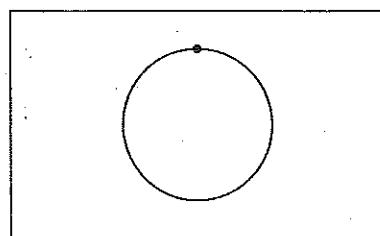
Problem 13 from the previous handout

A film runs through the projector at the rate of sixteen frames per second. On the screen, you see a moving car. In real life, the diameter of the car's tires is one meter. In the movie, the car's wheels rotate four times per second. What is the real life speed of the car?

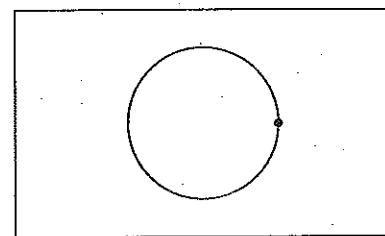
Here is the beginning of the solution. In the movie, the car's wheels rotate four times per second. The time period between the frames is

$$\tau = \frac{1}{16} \text{ sec.}$$

If during this time period of time in real life the wheels rotate n full times and a quarter, then we see the wheels rotating the right way on the screen.



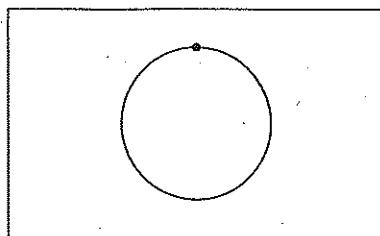
Frame 1.



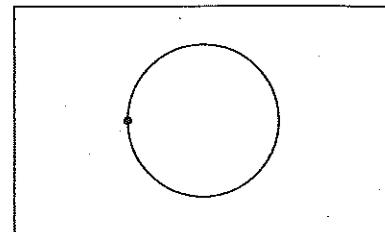
Frame 2

If the number of the rotations during the time period τ is n full times and three quarters, then we see the wheels rotating

the wrong way.



Frame 1



Frame 2

Please finish the solution.

Case 1. Rotates

$n = \frac{1}{4}$ times per frame -

$n = 0, 1, 2, 3, \dots$

$$\text{Then speed } v = \frac{x}{t} = \frac{\pi \cdot (n + \frac{1}{4})}{\frac{1}{16}} = \pi (16n + 4) \text{ meters per second.}$$

Use a calculator and
the conversion $1 \text{ m/s} = 2.237 \text{ mi/hr}$
to calculate:

n	$v (\text{m/s})$	$v (\text{mi/hr})$
0	12.57	28.11
1	62.83	140.6
2	113.10	253.0

Case 2. Rotates $n + \frac{3}{4}$

times per frame,

$n = 0, 1, 2, \dots$

$$\text{Then } v = \frac{x}{t} = \frac{\pi (n + \frac{3}{4})}{\frac{1}{16}} = \pi (16n + 3).$$

n	$v (\text{m/s})$	$v (\text{mi/hr})$
0	37.7	84.3
1	88.0	196.7
2	138.2	309.2
3		

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Among these six possibilities, only one is within the speed limit - 28.11 miles per hour. Since cars always go the speed limit in movies (truth value of this statement?) this must be the speed.