

PROOFS USING INDUCTION

MATH CIRCLE (HS1) 10/26/2014

Warmup Problems

1) Suppose you have an infinite chessboard (still colored black and white in a checkerboard pattern). Call a piece that can move in an extended “L” shape: 1 spot in any direction horizontally or vertically, and then n spots in a perpendicular direction an n -knight. (So for example, a normal knight is a 2-knight.)

What spaces on an infinite chessboard can a 1-knight reach? a 2-knight? a 3-knight? an n -knight? Try to prove it!

2) (Tower of Hanoi) Suppose you have 3 pegs (labelled A, B, C) and n disks of increasing size (labelled $1, 2, \dots, n$) which fit on the pegs. If multiple disks are on the same peg, they must be (from bottom to top) largest to smallest. You may move 1 disk at a time between peg (obeying the size restrictions). All the disks start on peg A . The goal is to move them all to peg B or C .

a) Show that (regardless of n) this is possible. Try to come up with a “best” method. How many moves does it take? Hint: Start with $n = 1$, then do $n = 2$, etc.

b) Try to prove your “best” method actually is best. That is, show that no one else can come up with a better method!

Note: Neither “I’ve seen this problem before and know this is best” nor “I’m pretty smart, and I can’t come up with anything better, so this must be best” are proofs!

Mathematical Induction (MI)

Mathematical Induction (MI) is a way to show that a property $P(n)$ holds for all $n = 1, 2, 3, \dots$. A proof using MI consists of two steps: (i) **Basis:** Show that $P(1)$ holds. (ii) **Induction:** Assume $P(k)$ holds, and show that $P(k + 1)$ holds. Then, MI tells us that $P(n)$ holds for all n .

For reference, copy down the proof of $1 + 2 + 2^2 + \dots + 2^{n-1} = 2^n - 1$ using MI presented in class.

Problems

0) Use MI to write out formal proofs for the Warmup Problems. (To save time you may reference things you have already proven.)

1) a) Prove $1 + 3 + 5 + \dots + (2n - 1) = n^2$.

b) Prove $1 + 2 + 3 + \dots + n = \frac{1}{2}n(n + 1)$.

Challenge) Prove $(1 + 2 + \dots + n)^2 = 1^3 + 2^3 + \dots + n^3$.

2) a) Prove $n^3 - n$ is a multiple of 6.

b) Prove $5^n - 1$ is a multiple of 4.

c) Prove $11^n - 4^n$ is a multiple of 7.

Challenge) Prove $2^{n+2} + 7^n$ is a multiple of 5.

3) a) Prove the number of diagonals in a convex n -gon is $n(n - 3)/2$.

b) Find, with proof, the sum of the measures of the interior angles in a convex n -gon.

4) Suppose we have a football league with n teams. Every team plays each other exactly once (and there are no ties). Prove that there is a way to number $(1, 2, \dots, n)$ so that team 1 beat team 2, 2 beat 3, etc.

5) Prove that a $2^n \times 2^n$ grid with any one box removed can be tiled (that is, covered with no overlaps) with L-shaped trominoes. (Note: The $n = 1$ case will let you know what a L-shaped tromino is!)

6) a) $2n$ dots are placed around the outside of the circle (half red, half blue). Prove it is possible to pick a starting place, so that if you travel clockwise around the circle, the number of red dots you have passed is at least the number of blue dots you have passed at all times.

b) A car is on a circular road that has n gas stations. The car starts with no gas, and the gas stations have exactly enough gas for the car to drive around the road once. Prove that the car can start at one of the gas stations and drive around the road once. Assume that the car's tank is large enough not to present a limitation.