Math Circle, HighSchool 2, 26 Oct 2014

October 26, 2014

1. (The Power of a Point) Let C be a circle, A, B, C, D points on the circle and let $AB \cap CD = P$. If T is also on the circle such that PT is a tangent, prove that $PA \cdot PB = PC \cdot PD = PT^2$. This number is called "the power of point P with respect to the circle C".

2. (Ptolemy's theorem) Let $ABCD$ be a cyclic quadrilateral. Then $AB \cdot CD + AD \cdot BC = AC \cdot BD$

3. In any regular penthagon, the ratio of the length of a diagonal to the length of a side is the golden ratio.

4. In $\triangle ABC$, angle B is right. Let $ACDE$ be a square drawn exterior to $\triangle ABC$. If M is the center of this square, find the measure of $\angle MBC$.

5. In $\triangle ABC$, let D be the midpoint of of AB, E the midpoint of side BC, and F the midpoint of AC. Let $k_1$be the circle passing through A, D,F; Let k_2 be the circle passing through B, E and D and let $k_3$be the circle through C,F and E. Prove that these three circles intersect at a point.

6. In $\triangle ABC$, $\angle BAC$ is right and $AB < AC$. If k is the circle with diameter AB, let E be the intersection of k with BC. Denote by t the tangent line to k that contains E and let D be the intersection of t with AC. Prove that $\triangle CDE$ is isosceles.

7. Circles $k_1$and $k_2$intersect at A and B. Line l is a common tangent to these circles, and it touches $k_1$at C and k_2 at D so that B is in interior of $\triangle ACD$. Prove that $\angle CAD + \angle CBD = 180$.

8. Let AB be a diameter of circle k. An arbitrary line l intersects k in P and Q. If $A_1$and $B_1$are the feet of the perpendiculars, respectively, from A and B to PQ, prove that $A_1P = B_1Q$.

9. In $\triangle ABC$, D and E are two points in the interior of side BC such that BD=CE and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.