LAMC Intermediate I & II October 19, 2014

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Warm-up

Problem 1 *Is the number N_{2012} of rectangles having integral side lengths and the perimeter 2012, greater, less, or equals to the number N_{2014} of rectangles having integral side lengths and the perimeter 2014? Please put the correct sign in the box below and provide the reason.*

$2x + 2y = 2012$

$x + y = 1006$

x	y
1	1005
2	1004
3	1003
⋮	⋮
1004	2
1005	1

1005 pairs ⇓ 1005 such rectangles

$2x + 2y = 2014$

$x + y = 1007$

x	y
1	1006
2	1005
3	1004
⋮	
1005	2
1006	1

1006 pairs ⟹ 1006 such rectangles

$$N_{2012} \boxed{<} N_{2014}$$

Problem 2 *Prove that among any six integers there exist two such that their difference is divisible by five.*

$\overline{} : \mathbb{Z} \longrightarrow \mathbb{Z}/5\mathbb{Z}$ assigns each integer to its equiv class mod 5. There are 5 such classes & 6 numbers, so 2 must map to same class $\Longrightarrow$ their difference is divisible by 5.

Problem 3 *Prove that among any $n+1$ integers there exist two such that their difference is divisible by n.*

Assign the $n+1$ numbers to the n congruence classes mod n. $n+1$ pigeons, n holes, so two in same congruence class. By definition, their difference is divisible by n.

Problem 4 *An old clock gets one minute behind time every hour. It is showing the correct time at the moment. When will it show the correct time next time?*

Must get 12 hours behind →

takes $1 \frac{\text{hr}}{\text{min}} \times \frac{60 \text{ min}}{\text{hr}} \times 12 \text{ hrs} = 720 \text{ hours}$

Problem 5 *The product of ~~two~~ 3 numbers equals* 1000. *The first two factors were increased 10%, the the third was decreased 20%. How did the product change?*

$xyz = 1000$

$(1.1x)(1.1y)(.8z) = (.968)xyz$

$= (.968)(1000) = 968$

$$\begin{array}{r} 1.1 \\ 1.1 \\ \hline 1.21 \\ .8 \\ \hline .968 \end{array}$$

Problem 6 *A shorter man's steps are 20% shorter than a taller man's steps, but the shorter man makes 20% more steps as he walks. Which of the men walks faster?*

$$V = \frac{d}{t} = \frac{d}{\text{step}} \cdot \frac{\text{step}}{t}$$

$$V_{short} = \left(\frac{.8 d_{tall}}{\text{step}_{tall}}\right)\left(\frac{1.2\ \text{step}_{tall}}{t_{tall}}\right) = (.8)(1.2) = .96\ V_{tall}$$

Tall man

Problem 7 *Alice, Bob, and Charlie ran a 100 meters race. When Alice finished, Bob was 10 meters behind her. When Bob finished, Charlie was 10 meters behind him. Assuming that they all ran at a constant speed, where was Charlie when Alice finished the race?*

$$V_{Bob} = .9\, V_{Alice}$$

$$V_{Charlie} = .9\, V_{Bob}$$

$$V_{Charlie} = (.9)(.9) V_{Alice} = .81\, V_{Alice}$$

19 meters behind Alice

Problem 8 *Two stores had equal prices. The first store decreased all the prices 10% one day and 10% the next day. The second store decreased all the prices 20% right away. What store (out of these two) would you go to?*

Store	A	B
Initial	P	P
After day 1	.9 P	P
After day 2	.9(.9P) = .81P	.8 P

Price

Store B

Problem 9 *The Pacific Ocean water has 3.5% of salt (with respect to weight). How much distilled water does one need to add to the Pasific Ocean water to reduce the amount of salt in the solution to 0.5%?*

$$3.5\% = \frac{\text{Salt}}{\text{Weight}_1} \qquad .5\% = \frac{\text{Salt}}{\text{Weight}_2}$$

$$.035 \text{ Weight}_1 = .005 \text{ Weight}_2$$

$$\text{Weight}_2 = 7 \text{ Weight}_1$$

Must add 6 × the weight of the pacific Ocean's worth of distilled water

Problem 10 *Find the smallest natural number n such that there exists a fraction m/n in between the numbers* 0.4 *and* 0.5.

n	possible decimals
1	–.0
2	–.0, –.5
3	–.$3\overline{33}$, –.$6\overline{66}$, –.0
4	–.0, –.25, –.5, –.75
5	–.2, –.4, –.6, –.8, –.0
6	–.0, –.$1\overline{6}$, –.$3\overline{3}$, –.5, –.~~45~~ $6\overline{6}$, –.$8\overline{3}$, –
7	–.0, –.14–, –.28–, –.42

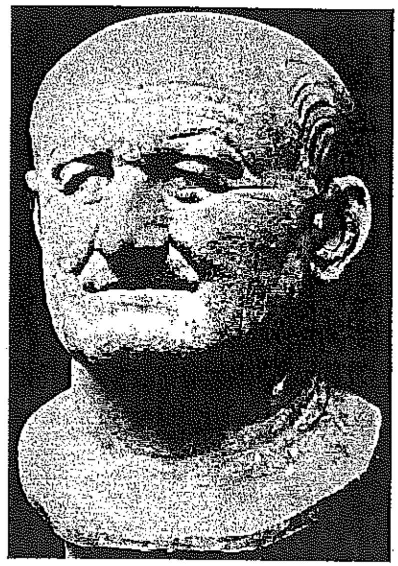
The following topic was suggested by one of our instructors, Eric Auld.

A Jewish general turned Roman historian, Josephus Flavius, lived in the first century AD. His book *The Judean War* describes a Jewish rebellion against the Romans that ended up with the destruction of Jerusalem in AD 70.

During the war, Josephus and his forty soldiers were trapped in a cave, with the Romans blocking the exit. The Jews chose suicide over capture. They decided to form a circle. Then a fighter would kill a neighboring fighter on his left with a blow of his sword and pass the sword further left until there was one man left standing. That one was supposed to kill himself. The last one happened to be Josephus who, instead of committing a suicide, turned himself over to the Romans.[1]

[1] We have slightly modified the story from the book to make it closer related to the subject of our interest, binary numbers.

Josephus proceeded to become a friend of the Roman emperor Vespasian Flavius (hence Josephus's Latin last name) and a popular historian.



Vespasian Flavius

The above story gave rise to a family of combinatorial problems all bearing the common name, the *Josephus Problem.*

first w/ the sword.

Problem 11 *What was the position of Josephus in the circle?*

19

18 left of 1st w/ sword.

Problem 12 *Suppose that there are n soldiers, including Josephus, in the cave. What should the position of Josephus be in order for him to stay alive? Hint: use binary numbers instead of the decimals.*

First, note that when $n=2^k$, the first person with the sword survives:

~~In the first time around, every even person dies. Then every person divisible by 4 dies, then 8, ... until every~~ ~~Then the first person gets the sword again since $2 \mid n$, and kills the 3rd person.~~

By induction on K. Trivial for $K=0$.

Now suppose true for some K. Then in the first round w/ $n=2^{K+1}$ people, exactly half the people die, and the first person gets the sword again. Therefore, we have a new circle with 2^K people, and the first person gets the sword again. ~~Therefore~~ By induction, the first person survives.

Now for ~~any~~ $n \neq 2^K$ *, let K be the greatest so* that $2^K < n$. Once ~~the~~ $n-2^K$ people die (which happens in round 1 by maximality of K), we get a circle with 2^K people. Thus, the person who gets the sword after $n-2^K$ deaths (person $2(n-2^K)+1$) survives.

If you are finished doing all the above, but there still remains some time ...

When you watch a moving car on the screen of a cinema theatre, sometimes the wheels of the vehicle seem to rotate the right way, but sometimes they seem to rotate the opposite way. Understanding this phenomenon is a part of solving the following problem.

Problem 13 *A film runs through the projector at the rate of sixteen frames per second. On the screen, you see a moving car. In real life, the diameter of the car's tires is one meter. In the movie, the car's wheels rotate four times per second. What is the real life speed of the car?*