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Warm-up

Problem 1 *A student was asked to divide some number by two and to add three to the result. By mistake, the student multiplied the number by two and subtracted three from the product. Accidentally, the student got the right answer. Find the original number.*

$$\frac{x}{2} + 3 = 2x - 3$$

$$2x - \frac{x}{2} = \frac{3}{2}x = 6$$

$$x = 4$$

Problem 2 *The Celsius temperature scale divides the temperature segment where the (distilled) water stays liquid into 100 equal parts. At the sea level, $0°C$ is the freezing point, $100°C$ is the boiling one. In the Fahrenheit scale, the (distilled) water freezes at $32°F$ and boils at $212°F$. The transition between the scales is given by the following formula.*

$$T°F = a \times T°C + b$$

- *Find a and b.*

$$32°F = a \times 0°C + b$$

$$b = 32°F$$

$$212°F = a \times 100°C + 32°F$$

$$a \times 100°C = 180°F$$

$$a = \frac{9}{5}\frac{°F}{°C}$$

- *What temperature is the same in both scales?*

$$T°F = T°C$$

$$T°C = a \times T°C + b$$

$$T°C(1-a) = b$$

$$T°C = \frac{b}{1-a} \overset{2}{=} \frac{32}{-\frac{4}{5}} = -\frac{5}{4} \cdot 32 = -40°$$

Problem 3 *Alice bought some chocolates and lollipops in the ratio 7:4. The price of all the chocolates to the price of all the lollipops was in the ratio 5:2. The girl spent \$127.40 in all. If each lollipop costs 15 cents less than each chocolate, how many lollipops did Alice buy?*

C = # of chocolates

L = # of lollipops

P_c = price / chocolate

P_L = price / lollipop

(1) $\frac{C}{L} = \frac{7}{4}$

(2) $\frac{P_c C}{P_L L} = \frac{5}{2}$

(3) $P_c C + P_L L = 127.4$

(4) $P_c = P_L + .15$

$$\frac{5}{2} = \frac{P_c C}{P_L L} = \frac{P_c}{P_L} \cdot \frac{7}{4}$$

$$\frac{P_c}{P_L} = \frac{20}{14} = \frac{10}{7}$$

$$P_L + .15 = P_c = \frac{10}{7} P_L$$

$$\frac{3}{7} P_L = .15$$

$$P_L = .35, \quad P_c = P_L + .15 = .35 + .15 = .$$

$$127.4 = P_c C + P_L L = P_c \cdot \frac{7}{4} L + P_L \cdot L$$

$$= \frac{1}{2} \cdot \frac{7}{4} L + .35 L = \frac{7}{8} L + .35 L = (.875 + .35) L = 1.225 L$$

$$L = \frac{127.4}{1.225} = 104$$

Binary numbers

Let us recall that there are only two digits in the *binary system,* 0 and 1. $0_{10} = 0_2$ (remember, the subscript denotes the base), $1_{10} = 1_2$, but $2_{10} = 10_2$, $3_{10} = 11_2$, and so on.

Example 1 *Find the binary representation of the number* 174_{10}.

Let us list all the powers of 2 *that are less than or equal to* 174.

n	2^n	n	2^n
0	1	5	32
1	2	6	64
2	4	7	128
3	8	8	256
4	16		

It turns out that the largest integral power of two still less *than* 174 *is* $128 = 2^7$.

$$174 = 128 + 46 = 2^7 + 46$$

The largest power of two less than 46 *is* $32 = 2^5$.

$$174 = 128 + 32 + 14 = 2^7 + 2^5 + 14$$

Finally, it is not hard to represent 14 *as a sum of powers of two,* $14 = 8 + 4 + 2$.

$$174 = 128 + 32 + 8 + 4 + 2 = 2^7 + 2^5 + 2^3 + 2^2 + 2^1$$

To write the number 174_{10} *in the binary form, we now need to fill the following eight boxes with either zeros or ones.*

$$\begin{array}{cccccccc} \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

The numbers under the boxes are the powers of two. If a power is absent from the decomposition of the number, then the corresponding box is filled with zero. For example, there is no $1 = 2^0$ *in the decomposition of the number* 174 *we have computed, so the first box from the right is filled with zero.*

$$\begin{array}{cccccccc} \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \underline{0} \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

$2 = 2^1$ *is present in the decomposition, so the box corresponding to the first power gets filled with one.*

$$\begin{array}{cccccccc} \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \underline{1} & \underline{0} \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}$$

$4 = 2^2$ is also present in the decomposition, so the box corresponding to the second power of two gets filled with one.

					1	1	0
7	6	5	4	3	2	1	0

Filling up all the boxes gives us the binary representation.

1	0	1	0	1	1	1	0
7	6	5	4	3	2	1	0

We write it down as follows.

$$174_{10} = 10101110_2$$

Problem 4 *Find the binary representations of the following decimal numbers.*

$12_{10} = 8+4 = 2^3+2^2 = 1100_2$

$25_{10} = 16+8+1 = 2^4+2^3+2^0 = 11001_2$

$32_{10} = 32 = 2^5 = 100000_2$

$100_{10} = 64+32+4 = 2^6+2^5+2^2$

$= 1100100_2$

Problem 5 *Find the decimal representations of the following binary numbers.*

$$101_2 = 2^0 + 2^2 = 1+4=5$$

$$11001_2 = 2^0 + 2^3 + 2^4 = 1+8+16=25$$

$$1000000_2 = 2^6 = 64$$

$$1010011_2 = 2^0+2^1+2^4+2^6 = 1+2+16+64=83.$$

Problem 6 *Complete the binary addition and multiplication tables below.*

$+$	0	1
0	0	1
1	1	10

$\times$	0	1
0	0	0
1	0	1

Problem 7 *Use long addition to sum up the following two binary numbers without switching to the decimals.*

$$\begin{array}{ccccccc} & & & 1 & 1 & 1 & \\ & 1 & 1 & 0 & 1 & 1 & 1 \\ + & & 1 & 0 & 0 & 1 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}$$

Then find the decimal representation of the summands and of the sum and check your answer.

$$110111_2 = 2^0 + 2^1 + 2^2 + 2^4 + 2^5 = 1 + 2 + 4 + 16 + 32$$

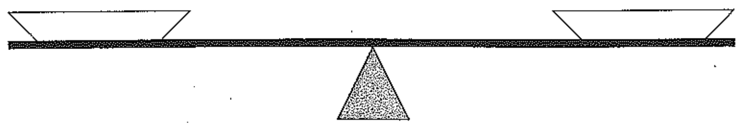
$$= 55$$

$$10011_2 = 2^0 + 2^1 + 2^4 = 1 + 2 + 16 = 19$$

$$55 + 19 = 74$$

$$1001010 = 2^1 + 2^3 + 2^6 = 2 + 8 + 64 = 74 \quad \checkmark$$

A balance scale is a device for comparing weights very similar to a see-saw at a children's playground. You put two weights on the scales' plates. If the weights are equal, the scales remain in balance. If the weights are different, the lighter weight goes up.



Problem 8 *Given a balance scale and the weights of 1 lb, 2 lbs, 4 lbs, and 8 lbs (one of each), prove that you can weigh any (integral) load from 1 to 15 lbs. Why do you think this is possible?*

Each of the numbers 1, 2, ..., 15 has a a four digit binary representation, and thus can be written as a sum of 0001, 0010, 0100, 1000. Ie - 1, 2, 4, 8.

Problem 9 *Perform the following long multiplication without switching to the decimals.*

$$
\begin{array}{r}
11011 \\
\times \quad 1010 \\
\hline
0 \\
110110 \\
0 \\
+\ 11011000 \\
\hline
100001110
\end{array}
$$

Then find the decimal representations of the factors and of the product and check your answer.

$$11011_2 = 2^0 + 2^1 + 2^3 + 2^4 = 1+2+8+16 = 27$$

$$1010_2 = 2^1 + 2^3 = 2+8 = 10$$

$$10 \cdot 27 = 270$$

$$100001110 = 2^1 + 2^2 + 2^3 + 2^8$$
$$= 2+4+8+256 = 270. \checkmark$$

Problem 10 *Find the missing binary digits.*

$$
\begin{array}{rccccc}
 & & 1 & \underline{0} & 0 & 1 \\
+ & & & \underline{1} & \underline{1} & 1 \\
\hline
 & 1 & 0 & 0 & 0 & 0
\end{array}
\qquad\qquad
\begin{array}{rccccc}
 & & 1 & \underline{1} & 0 & 1 \\
+ & & & 1 & 0 & \underline{1} \\
\hline
 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

$$
\begin{array}{rccc}
 & 1 & \underline{0} & 1 \\
\times & & & \underline{1} \\
\hline
 & 1 & 0 & 1
\end{array}
\qquad\qquad
\begin{array}{rcccc}
 & & & 1 & \underline{1} \\
\times & & & \underline{1} & 1 \\
\hline
 & 1 & 0 & 0 & 1
\end{array}
$$

Problem 11 *Recover the missing binary digits making the below inequalities correct.*

$$1\ 0\ 0\ \underline{1}\ \underline{1} \; > \; 1\ 0\ 0\ 1\ 0$$

$$1\ 0\ \underline{1}\ \underline{1}\ 0 \; > \; 1\ 0\ 1\ 0\ 0$$

$$1\ \underline{1}\ 0\ 1\ \underline{1} \; > \; 1\ 1\ 0\ 1\ 0$$

Problem 12 *Perform the following subtraction of the binary numbers.*

$$\begin{array}{r} 10 \\ \cancel{1}\ 1\cancel{0}\ 1\cancel{0}\ \cancel{1}\ 10 \\ -\quad 1\ \ 0\ \ 1\ \ 1 \\ \hline 1\ \ 1\ \ 1 \end{array}$$

Then find the decimal representations of the numbers and of the difference and check your answer.

$$10010_2 = 2^1 + 2^4 = 2 + 16 = 18$$

$$1011_2 = 2^0 + 2^1 + 2^3 = 1 + 2 + 8 = 11$$

$$18 - 11 = 7$$

$$111 = 2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7. \ \checkmark$$

Problem 13 *Solve the following equations in the binaries.*

$$x + 11 = 1\cancel{1}01 \qquad x = 1010$$

$$\begin{array}{r} 1101 \\ -\ 11 \\ \hline 1010 \end{array}$$

$$x - 10 = 101 \qquad x = 111$$

$$x - 1101 = 11011 \qquad x = 101000$$

$$\begin{array}{r} 11011 \\ +\ 1101 \\ \hline 101000 \end{array}$$

$$x + 1110 = \cancel{1}\cancel{0}\cancel{0}01 \qquad x = 11$$

$$\begin{array}{r} 10001 \\ -\ 1110 \\ \hline 00011 \end{array}$$

$$x + 111 = 1\cancel{1}\cancel{1}\cancel{1}0 \qquad x = 10111$$

$$\begin{array}{r} 01010 \\ -\ 111 \\ \hline 10111 \end{array}$$

$$x - 1001 = 11101 \qquad x = 100110$$

$$\begin{array}{r} 11101 \\ 1001 \\ \hline 100110 \end{array}$$

Complement of a decimal number

A *decimal complement* of a positive decimal integer x is a positive decimal integer c such that $x + c$ equals the smallest number of the form 10^n greater than or equal to x. For example, the decimal complement of 7 is 3 $(7 + 3 = 10)$.

Let us find the decimal complement of 243. The smallest power of ten greater than the number is 1000. To have zero as the last digit of the sum, the last digit of the complement must be equal to 7.

$$c = \underline{}\ \underline{}\ \underline{7}$$

$3 + 7 = 101$, so one carries over. Therefore, the second digit of the complement must be equal to five.

$$c = \underline{}\ \underline{5}\ \underline{7}$$

$1 + 4 + 5 = 10$, so one carries over again. Hence, the third digit of c must be equal to seven.

$$c = \underline{7}\ \underline{5}\ \underline{7}$$

The number c is the desired complement.

$$243 + 757 = 1000$$

Problem 14 *Find decimal complements of the following decimal numbers.*

x	c
24	76
179	821
7,834	2,166

A decimal complement is a great computational tool that replaces a trickier operation, subtraction, with an easier one, addition. For example, let us see how it helps us to subtract 7,834 from 9,632.

$$9,632-7,834 = 9,632-(10,000-2,166) = 9,632+2,166-10,000$$

Instead of subtracting 7,834 from 9,632, we add the decimal complement of 7,834, the number 2,166, to 9,632 and then subtract 10,000.

$$9,632-7,834 = 9,632+2,166-10,000 = 11,798-10,000 = 1,798$$

Problem 15 *Use the above trick to solve the following subtraction problems.*

$92 - 24 =$ $92-(100-76) = 92+76-100 = 168-100=68$

$533 - 179 =$ $533+821-1000 = 1354$

$1025 - 787 =$ $1025+213-1000 = 1238-1000=238$

Complement of a binary number

A *binary complement* of a positive binary integer x is a positive binary integer c such that $x + c$ equals the smallest number of the form 2^n greater than or equal to x. For example, the decimal complement of 101 is 11 $(101 + 11 = 1000)$. The following algorithm describes how to construct c from x.

1. Read x from the right to the left.

2. If the first digit on the right is zero, move on to the next digit.

3. Do not change the first one you meet on the way.

4. After the first one, change all the zeros you meet to ones and all the ones to zeros.

Let us use the algorithm to find the binary complement of the number $x = 10101100$.

The algorithm tells us to keep two zeros and the first one on the right hand side of x.

$$c = \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\ 1\ }\ \underline{\ 0\ }\ \underline{\ 0\ }$$

From this point on, we need to switch zeros and ones. So the next one becomes zero.

$$c = \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\ 0\ }\ \underline{\ 1\ }\ \underline{\ 0\ }\ \underline{\ 0\ }$$

The next zero becomes one.

$c = \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}$

And so froth to the end.

$c = \underline{0}\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{1}\ \underline{0}\ \underline{0}$

Dropping the zero at the front of the complement, we get the following number.

$$c = 1010100$$

Problem 16 *Use long addition to check that c is indeed the complement of x.*

$$\begin{array}{r} 1\,0\,1\,0\,1\,1\,0\,0 \\ +\quad 1\,0\,1\,0\,1\,0\,0 \\ \hline 1\,0\,0\,0\,0\,0\,0\,0\,0 \end{array}$$

What power of two do x and c add up to?

$n =$ 8

Problem 17 *Find binary complements of the following numbers.*

$x = 111$ $c =$ 1

$x = 1100$ $c =$ 100

Similar to the decimal case, we can use a binary complement to make the operation of subtraction simpler. Suppose that we need to subtract a binary number x from a binary number y. Since the sum of x and its complement c equals some power of two, $x + y = 2^n$, we can perform the subtraction as follows.

$$y - x = y - (2^n - c) = y + c - 2^n$$

This way we replace a general subtraction problem by an addition problem followed by a subtraction of the simplest possible kind, that of a number having a one for the first digit and zeros for the rest of them.

To show the efficiency of the method, let us use it to solve Problem 12. We need to subtract 1011 from 10010. Let us rewrite the greater number first. Instead of 1011, let us write its binary complement we are about to add. The complement is found using the algorithm on page 17. Finally, let us write down the sum of 1011 and its complement that we will subtract.

$$\begin{array}{cccccc}
 & 1 & 0 & 0 & 1 & 0 \\
- & & 1 & 0 & 1 & 1 \\
\hline
 & 1 & 0 & 0 & 1 & 0 \\
+ & & 0 & 1 & 0 & 1 \\
- & 1 & 0 & 0 & 0 & 0 \\
\hline
\end{array}$$

Performing first the addition and then the subtraction finishes the computation.

$$\begin{array}{rrrrrr} & 1 & 0 & 0 & 1 & 0 \\ - & & 1 & 0 & 1 & 1 \\ \hline & 1 & 0 & 0 & 1 & 0 \\ + & & 0 & 1 & 0 & 1 \\ - & 1 & 0 & 0 & 0 & 0 \\ \hline & 1 & 0 & 1 & 1 & 1 \\ - & 1 & 0 & 0 & 0 & 0 \\ \hline & & & 1 & 1 & 1 \end{array}$$

Question 1 *It may be that the new subtraction algorithm is a bit easier than the old one, but do we really need it?*

We don't, but our computers do. The heart of the computer is a chip known as a CPU, central processing unit. Nearly all it can do is adding binary numbers. You just have learned how it does subtraction!

Problem 18 *Use the new method to solve the following subtraction problems.*

$$\begin{array}{r} 10101 \\ -\quad 110 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 11101 \\ -\quad 1011 \\ \hline \end{array}$$

$$\begin{array}{r} 10101 \\ +\quad 10 \\ -1000 \\ \hline 10111 \\ -1000 \\ \hline 1111 \end{array} \qquad\qquad \begin{array}{r} 11101 \\ +\quad 101 \\ -10000 \\ \hline 1101 \\ +101 \\ \hline 10010 \end{array}$$

If you are finished doing all the above, but there still remains some time ...

Problem 19 *Three farmers went to the market to sell chicken. One farmer brought 10 chicken to sell, the second brought 16, while the third brought 26. The farmers agreed to charge the same price for their animals. After the lunch break, afraid that they may not sell all of the chicken, the farmers lowered the price. As a result, they sold all the chicken, making \$35 each. What was the price before and after the lunch break?*

Let x_1 = chickens sold by farmer 1 before break
x_2 = " " after "
y_1 = " " farmer 2 before "
y_2 = " " " after "
z_1 = " " farmer 3 before "
z_2 = " " " after ", and let
p be the price before break, q the price after break.

Then get a system:

$$x_1 p + x_2 q = 35 \qquad x_1 + x_2 = 10$$
$$y_1 p + y_2 q = 35 \qquad y_1 + y_2 = 16$$
$$z_1 p + z_2 q = 35 \qquad z_1 + z_2 = 26.$$

Combining, we get

$$x_1 p + (10 - x_1)q = 35$$
$$y_1 p + (16 - y_1)q = 35$$
$$z_1 p + (26 - z_1)q = 35,$$

or
(1) $x_1(p-q) + 10q = 35$
(2) $y_1(p-q) + 16q = 35$
(3) $z_1(p-q) + 26q = 35$

(2)−(1) gives: $(y_1 - x_1)(p-q) + 6q = 0$
(3)−(2) gives: $(z_1 - y_1)(p-q) + 10q = 0$ (*)
(3)−(1) gives: $(z_1 - x_1)(p-q) + 16q = 0$ (**). Checking signs, we know ~~that~~ are all >0, as ~~one~~ is $p-q$, so that ~~...~~ are all negative. ~~Now divide (**) by (*) to get~~ $\cancel{\frac{z_1 - x_1}{z_1 - y_1}}$ Then (*) and (**) can become $(y_1 - z_1)(p-q) = 10q$ and $(x_1 - z_1)(p-q) = 16q$, with all ~~...~~ terms positive.
Divide the second by the first to get $\frac{x_1 - z_1}{y_1 - z_1} = \frac{16}{10} = \frac{8}{5}$. Ie. $5(x_1 - z_1) = 8(y_1 - z_1)$.

Then $10 \geq x_1 \geq x_1 - z_1$, and $8 \mid (x_1 - z_1)$ since $\gcd(8,5) = 1$. ~~Thus~~ Since also, $x_1 - z_1 > 0$, this means we must have $x_1 - z_1 = 8$, which immediately gives $y_1 - z_1 = 5$. We now list the possibilities for x_1, y_1, z_1 as follows:

x_1	y_1	z_1
8	5	0
9	6	1

which are the only possibilities because $z_1 \geq 0$ and $x_1 \leq 10$. We can then plug each of these possibilities into the original system to find p and q. The only case in which ... representation of p, q terminating ... 100ths is the second ... $p = 3.7$...

Homework

An *algorithm* is a clear finite set of instructions needed to perform a computation or to solve a problem. For example, the algorithm you have seen on page 17 prescribes the steps needed to efficiently execute the subtraction of binary numbers.

Problem 20 *Design an algorithm for the division of binary numbers. Hint: realize division as repeated subtraction.*

We want, given (binary) numbers n and m, to find (binary) numbers q and r so that $n = qm + r$, with $0 \le r < m$, and without converting to decimal numbers.

(Using same method as p. 11)

So <u>Step 1</u>. Check if $0 \le n < m$. If so, $q = 0$, $r = n$, and the process stops. If not $\longrightarrow$

<u>Step 2</u>. Use the algorithm on page 19 to subtract m from n. If $0 \le n-m < m$ then $q = 1$ and $r = n-m$, and we stop. If not, go to step 3.

<u>Step 3</u>. Subtract m from $n-m$ and test whether $0 \le n-2m < m$ is true. If so, stop – $q = 2$, $r = n-2m$. If not, continue to step 4.

$\vdots$

<u>Step n</u>. Subtract m from $n-(n-2)m$, and test whether the result satisfies $0 \le n-(n-1)m < m$. If so, $q = n-1$ and $r = m-(n-1)m$. If not, continue to step $n+1$.

<u>Note</u>. The process must stop since (n is finite, and) the number being subtracted