PROOFS REVISITED

MATH CIRCLE (HS) 10/19/2014

Recall the two types of proofs introduced last time: 1) *Direct Proofs*: Prove the actual result of the thoerem, 2) *Proofs by Contradiction*: Show that the opposite/negation of the result is impossible (and hence the result must hold).

Warmup

a) There are 8 people at a party. Prove it is possible for each person to shake hands with exactly 3 others.

b) There are 9 people at a party. Prove it is impossible for each person to shake hands with exactly 3 others.

Extra) There are n people at a party. Prove it is possible or impossible for each person to shake hands with exactly 3 others.

Problems

For each of the following:

i) Write out the negation of the statement you are trying to prove.

ii) Prove the statement and state whether you are doing a Direct Proof or a Proof by Contradiction.

iii) Try to prove the statement with the other method, or explain why you think the other method is not feasible.

1) a) The sum of two rational numbers is rational.

b) The sum of two irrational numbers is not necessarily irrational.

c) The product of a non-zero rational number and an irrational number is irrational.

2) Let a, b, c be positive integers. Suppose a divides b and b divides c. Then a divides c.

3) Suppose n is an integer. If n^2 is even, then n is even.

4) a) If $(a+b)^2 = a^2 + b^2$ for all real numbers b, then a must be zero.

b) How is the fact that we assume the equality holds for *all* b used in your proof?

This handout is based on problems from Z. Stankova and T. Rike "A Decade of the Berkeley Math Circle."

Contrapositive

Note that both 3) and 4) have the form "If ___ then ___", or using variables, "If P then Q". When proving both by contradiction (i.e. by assuming not Q, written $\sim Q$), the contradiction you got was probably $\sim P$. That is, instead of proving "If P then Q", you proved "If $\sim Q$ then $\sim P$". This idea works in general, and is called a *Proof by Contrapositive.*

Prove the following using a Proof by Contrapositive.

5) If no angles in a quadrilateral are obtuse, then the quadrilateral is a rectangle.

6) If a and b are integers with b odd, then $+1, -1$ are not roots of $ax^4 + bx^2 + a$.

Extra Problems

General Hint: All the problems below have fairly short proofs, IF you organize them correctly.

7) The length of any chord on a circle has length less than or equal to the diameter of the circle.

8) Suppose you have an infinite chessboard (still colored black and white in a checkerboard pattern).

a) Show that a knight (which moves in an "L" shape: 1 spot in any direction horizontally or vertically, and then 2 spots in a perpendicular direction) can reach any position on the board.

b) Call a piece that can move in an extended "L" shape: 1 spot in any direction spot in any direction horizontally or vertically, and then n spots in a perpendicular direction an n-knight. (So for example, a normal knight is a 2-knight.) What spaces on an infinite chessboard can a 3-knight reach?

Extra) What spaces on an infinite chessboard can an n-knight reach, for $n = 1, 2, 3, 4, 5, \ldots$? Try to prove your answer! Hint: Work your way up, and see if you can see a pattern.