Highschool 2 Math Circle, Oct 12

1. A quadratic polynomial $f(x) = ax^2+bx+c$ is such that its values $f\left(\frac{1}{a}\right)$ and $f(c)$ have opposite signs. Show that the roots of the polynomial have opposite signs.

2. A store is selling 21 white shirts and 21 purple shirts which are all hanging on hangers in a row. Find the smallest k such that for any initial order of the shirts you can take out k white shirts and k purple shirts in such a way that the remaining white shirts are all next to each other and the remaining purple shirts are all next to each other as well.

3. Let M be the midpoint of the side AC of a triangle $\triangle ABC$. Let P be the midpoint of CM. Let Q be the point of intersection of the circle circumscribed around $\triangle ABP$ and the segment BC, and such that Q lies inside of $\triangle ABC$. Prove that $\angle ABM = \angle MQP$.

4. Several white and black points are marked on the plane. There is an arrow from each white point to each black point. Moreover, each arrow is marked with a natural number.
   For any closed path the product of the numbers on the arrows that are traversed in the correct direction equals to the product of the numbers on the arrows that are traveresed in the opposite direction.
   Is it true that one can assign a number to each point in such a way that the number assigned to an arrow is the product of numbers written on the arrow's ends?

5. A polynomial $P(x)$ is such that $P(0) = 1$ and
   $$(P(x))^2 = 1 + x + x^{100} \cdot Q(x),$$
   where $Q(x)$ is a polynomial. Show that the coefficient of x^{99} in the polynomial $(P(x) + 1)^{100}$ is 0.

6. Find all values of a for which there exist the values x, y, z for which both the numbers

$$\cos x,\ \cos y,\ \cos z$$

and the numbers

$$\cos(x+a),\ \cos(y+a),\ \cos(z+a)$$

form an arithmetic progressions.

7. Let $ABCD$ be a parallelogram, and P and Q be points on the sides AD and CD respectively so that $\angle AOP = \angle COQ = \angle ABC$.

   (a) Show that $\angle ABP = \angle CBQ$.

   (b) Show that the lines AQ and CP intersect each other at a point which lies on the circumscribed circle of $\triangle ABC$.

8. Sasha discovered that exactly n buttons on his pocket calculator work. It also turned out that any whole number from 1 to $99,999,999$ can either be inputted using the working buttons, or can be obtained as a sum of two whole numbers that can be inputted using the working buttons. What is the smalles possible value of n?

9. Let $a, b, c > 0$ such that $abc = 1$. Prove that $\frac{1}{1+a+b} + \frac{1}{1+b+c} + \frac{1}{1+c+a} \leq 1$.