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Warm-up

Problem 1 *A student was asked to divide some number by two and to add three to the result. By mistake, the student multiplied the number by two and subtracted three from the product. Accidentally, the student got the right answer. Find the original number.*

Problem 2 *The Celsius temperature scale divides the temperature segment where the (distilled) water stays liquid into 100 equal parts. At the sea level, 0°C is the freezing point, 100°C is the boiling one. In the Fahrenheit scale, the (distilled) water freezes at 32°F and boils at 212°F . The transition between the scales is given by the following formula.*

$$T^\circ F = a \times T^\circ C + b$$

- Find a and b .
- What temperature is the same in both scales?

Problem 3 *Alice bought some chocolates and lollipops in the ratio 7:4. The price of all the chocolates to the price of all the lollipops was in the ratio 5:2. The girl spent \$127.40 in all. If each lollipop costs 15 cents less than each chocolate, how many lollipops did Alice buy?*

Binary numbers

Let us recall that there are only two digits in the *binary system*, 0 and 1. $0_{10} = 0_2$ (remember, the subscript denotes the base), $1_{10} = 1_2$, but $2_{10} = 10_2$, $3_{10} = 11_2$, and so on.

Example 1 Find the binary representation of the number 174_{10} .

Let us list all the powers of 2 that are less than or equal to 174.

n	2^n	n	2^n
0	1	5	32
1	2	6	64
2	4	7	128
3	8	8	256
4	16		

It turns out that the largest integral power of two still less

than 174 is $128 = 2^7$.

$$174 = 128 + 46 = 2^7 + 46$$

The largest power of two less than 46 is $32 = 2^5$.

$$174 = 128 + 32 + 14 = 2^7 + 2^5 + 14$$

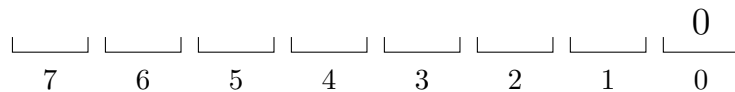
Finally, it is not hard to represent 14 as a sum of powers of two, $14 = 8 + 4 + 2$.

$$174 = 128 + 32 + 8 + 4 + 2 = 2^7 + 2^5 + 2^3 + 2^2 + 2^1$$

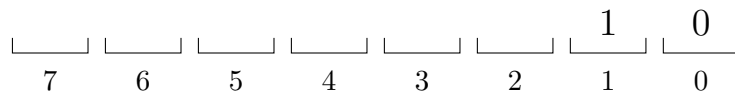
To write the number 174_{10} in the binary form, we now need to fill the following eight boxes with either zeros or ones.



The numbers under the boxes are the powers of two. If a power is absent from the decomposition of the number, then the corresponding box is filled with zero. For example, there is no $1 = 2^0$ in the decomposition of the number 174 we have computed, so the first box from the right is filled with zero.



$2 = 2^1$ is present in the decomposition, so the box corresponding to the first power gets filled with one.



$4 = 2^2$ is also present in the decomposition, so the box corresponding to the second power of two gets filled with one.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & 1 & 1 & 0 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}$$

Filling up all the boxes gives us the binary representation.

$$\begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ \hline 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ \hline \end{array}$$

We write it down as follows.

$$174_{10} = 10101110_2$$

Problem 4 Find the binary representations of the following decimal numbers.

$$12_{10} =$$

$$25_{10} =$$

$$32_{10} =$$

$$100_{10} =$$

Problem 5 Find the decimal representations of the following binary numbers.

$$101_2 =$$

$$11001_2 =$$

$$1000000_2 =$$

$$1010011_2 =$$

Problem 6 Complete the binary addition and multiplication tables below.

+	0	1
0		
1		

×	0	1
0		
1		

Problem 7 Use long addition to sum up the following two binary numbers without switching to the decimals.

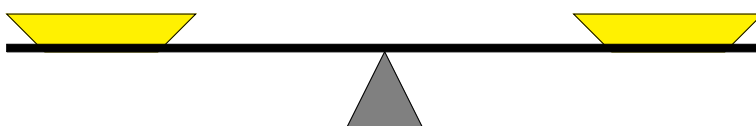
$$\begin{array}{r}
 1\ 1\ 0\ 1\ 1\ 1 \\
 +\quad 1\ 0\ 0\ 1\ 1 \\
 \hline
 \end{array}$$

Then find the decimal representation of the summands and of the sum and check your answer.

$$110111_2 =$$

$$10011_2 =$$

A balance scale is a device for comparing weights very similar to a see-saw at a children's playground. You put two weights on the scales' plates. If the weights are equal, the scales remain in balance. If the weights are different, the lighter weight goes up.



Problem 8 *Given a balance scale and the weights of 1 lb, 2 lbs, 4 lbs, and 8 lbs (one of each), prove that you can weigh any (integral) load from 1 to 15 lbs. Why do you think this is possible?*

Problem 9 *Perform the following long multiplication without switching to the decimals.*

$$\begin{array}{r} 11011 \\ \times 1010 \\ \hline \end{array}$$

Then find the decimal representations of the factors and of the product and check your answer.

$$11011_2 =$$

$$1010_2 =$$

Problem 10 Find the missing binary digits.

$$\begin{array}{r}
 1 \quad \square \quad 0 \quad 1 \\
 + \quad \square \quad \square \quad 1 \\
 \hline
 1 \quad 0 \quad 0 \quad 0 \quad 0
 \end{array}$$

$$\begin{array}{r}
 1 \quad \square \quad 0 \quad 1 \\
 + \quad 1 \quad 0 \quad \square \\
 \hline
 1 \quad 0 \quad 0 \quad 1 \quad 0
 \end{array}$$

$$\begin{array}{r}
 1 \quad \square \quad 1 \\
 \times \quad \square \\
 \hline
 1 \quad 0 \quad 1
 \end{array}$$

$$\begin{array}{r}
 1 \quad \square \\
 \times \quad \square \quad 1 \\
 \hline
 1 \quad 0 \quad 0 \quad 1
 \end{array}$$

Problem 11 Recover the missing binary digits making the below inequalities correct.

$$1 \ 0 \ 0 \ \square \ \square \ > \ 1 \ 0 \ 0 \ 1 \ 0$$

$$1 \ 0 \ \square \ \square \ 0 \ > \ 1 \ 0 \ 1 \ 0 \ 0$$

$$1 \ \square \ 0 \ 1 \ \square \ > \ 1 \ 1 \ 0 \ 1 \ 0$$

Problem 12 *Perform the following subtraction of the binary numbers.*

$$\begin{array}{r} 1\ 0\ 0\ 1\ 0 \\ -\quad 1\ 0\ 1\ 1 \\ \hline \end{array}$$

Then find the decimal representations of the numbers and of the difference and check your answer.

$$10010_2 =$$

$$1011_2 =$$

Problem 13 *Solve the following equations in the binaries.*

$$x + 11 = 1101 \qquad x =$$

$$x - 10 = 101 \qquad x =$$

$$x - 1101 = 11011 \qquad x =$$

$$x + 1110 = 10001 \qquad x =$$

$$x + 111 = 11110 \qquad x =$$

$$x - 1001 = 11101 \qquad x =$$

Complement of a decimal number

A *decimal complement* of a positive decimal integer x is a positive decimal integer c such that $x + c$ equals the smallest number of the form 10^n greater than or equal to x . For example, the decimal complement of 7 is 3 ($7 + 3 = 10$).

Let us find the decimal complement of 243. The smallest power of ten greater than the number is 1000. To have zero as the last digit of the sum, the last digit of the complement must be equal to 7.

$$c = \underline{\quad} \underline{\quad} \underline{7}$$

$3 + 7 = 10$, so one carries over. Therefore, the second digit of the complement must be equal to five.

$$c = \underline{\quad} \underline{5} \underline{7}$$

$1 + 4 + 5 = 10$, so one carries over again. Hence, the third digit of c must be equal to seven.

$$c = \underline{7} \underline{5} \underline{7}$$

The number c is the desired complement.

$$243 + 757 = 1000$$

Problem 14 Find decimal complements of the following decimal numbers.

x	c
24	
179	
7,834	

A decimal complement is a great computational tool that replaces a trickier operation, subtraction, with an easier one, addition. For example, let us see how it helps us to subtract 7,834 from 9,632.

$$9,632 - 7,834 = 9,632 - (10,000 - 2,166) = 9,632 + 2,166 - 10,000$$

Instead of subtracting 7,834 from 9,632, we add the decimal complement of 7,834, the number 2,166, to 9,632 and then subtract 10,000.

$$9,632 - 7,834 = 9,632 + 2,166 - 10,000 = 11,798 - 10,000 = 1,798$$

Problem 15 *Use the above trick to solve the following subtraction problems.*

$$92 - 24 =$$

$$533 - 179 =$$

$$1025 - 787 =$$

Complement of a binary number

A *binary complement* of a positive binary integer x is a positive binary integer c such that $x + c$ equals the smallest number of the form 2^n greater than or equal to x . For example, the decimal complement of 101 is 11 ($101 + 11 = 1000$). The following algorithm describes how to construct c from x .

1. Read x from the right to the left.
2. If the first digit on the right is zero, move on to the next digit.
3. Do not change the first one you meet on the way.
4. After the first one, change all the zeros you meet to ones and all the ones to zeros.

Let us use the algorithm to find the binary complement of the number $x = 10101100$.

The algorithm tells us to keep two zeros and the first one on the right hand side of x .

$$c = \boxed{} \boxed{} \boxed{} \boxed{} \boxed{} \boxed{1} \boxed{0} \boxed{0}$$

From this point on, we need to switch zeros and ones. So the next one becomes zero.

$$c = \boxed{} \boxed{} \boxed{} \boxed{} \boxed{0} \boxed{1} \boxed{0} \boxed{0}$$

The next zero becomes one.

$$c = \boxed{} \boxed{} \boxed{} \boxed{1} \boxed{0} \boxed{1} \boxed{0} \boxed{0}$$

And so forth to the end.

$$c = \boxed{0} \boxed{1} \boxed{0} \boxed{1} \boxed{0} \boxed{1} \boxed{0} \boxed{0}$$

Dropping the zero at the front of the complement, we get the following number.

$$c = 1010100$$

Problem 16 Use long addition to check that c is indeed the complement of x .

$$\begin{array}{r} 1\ 0\ 1\ 0\ 1\ 1\ 0\ 0 \\ +\quad 1\ 0\ 1\ 0\ 1\ 0\ 0 \\ \hline \end{array}$$

What power of two do x and c add up to?

$$n =$$

Problem 17 Find binary complements of the following numbers.

$$x = 111$$

$$c =$$

$$x = 1100$$

$$c =$$

Similar to the decimal case, we can use a binary complement to make the operation of subtraction simpler. Suppose that we need to subtract a binary number x from a binary number y . Since the sum of x and its complement c equals some power of two,

$$x + y = 2^n = 1 \underbrace{0 \dots 0}_n 2,$$

we can perform the subtraction as follows.

$$y - x = y - (2^n - c) = y + c - 2^n$$

This way we replace a general subtraction problem by an addition problem followed by a subtraction of the simplest possible kind, that of a number having a one for the first digit and zeros for the rest of them.

To show the efficiency of the method, let us use it to solve Problem 12. We need to subtract 1011 from 10010. Let us rewrite the greater number first. Instead of 1011, let us write its binary complement we are about to add. The complement is found using the algorithm on page 17. Finally, let us write down the sum of 1011 and its complement that we will subtract.

$$\begin{array}{r}
 1 \ 0 \ 0 \ 1 \ 0 \\
 - \quad 1 \ 0 \ 1 \ 1 \\
 \hline
 1 \ 0 \ 0 \ 1 \ 0 \\
 + \quad 0 \ 1 \ 0 \ 1 \\
 - \ 1 \ 0 \ 0 \ 0 \ 0 \\
 \hline
 \end{array}$$

Performing first the addition and then the subtraction completes the computation.

$$\begin{array}{r}
 1\ 0\ 0\ 1\ 0 \\
 -\quad 1\ 0\ 1\ 1 \\
 \hline
 1\ 0\ 0\ 1\ 0 \\
 +\quad 0\ 1\ 0\ 1 \\
 -\quad 1\ 0\ 0\ 0\ 0 \\
 \hline
 1\ 0\ 1\ 1\ 1 \\
 -\quad 1\ 0\ 0\ 0\ 0 \\
 \hline
 1\ 1\ 1
 \end{array}$$

Question 1 *It may be that the new subtraction algorithm is a bit easier than the old one, but do we really need it?*

We don't, but our computers do. The heart of the computer is a chip known as a CPU, central processing unit. Nearly all it can do is adding binary numbers. You just have learned how it does subtraction!

Problem 18 *Use the new method to solve the following subtraction problems.*

$$\begin{array}{r} 1\ 0\ 1\ 0\ 1 \\ -\quad\quad 1\ 1\ 0 \\ \hline \end{array}$$

$$\begin{array}{r} 1\ 1\ 1\ 0\ 1 \\ -\quad\quad 1\ 0\ 1\ 1 \\ \hline \end{array}$$

If you are finished doing all the above, but there still remains some time ...

Problem 19 *Three farmers went to the market to sell chicken. One farmer brought 10 chicken to sell, the second brought 16, while the third brought 26. The farmers agreed to charge the same price for their animals. After the lunch break, afraid that they may not sell all of the chicken, the farmers lowered the price. As a result, they sold all the chicken, making \$35 each. What was the price before and after the lunch break?*

Homework

An *algorithm* is a clear finite set of instructions needed to perform a computation or to solve a problem. For example, the algorithm you have seen on page 17 prescribes the steps needed to efficiently execute the subtraction of binary numbers.

Problem 20 *Design an algorithm for the division of binary numbers. Hint: realize division as repeated subtraction.*