SEQUENCES AND SERIES

MATH CIRCLE (HS1) 3/2/2014

Recall from last time that a sequence is an ordered list of elements. It can be either finite ($\langle a_0, a_1, \ldots, a_{l-1}\rangle$) or infinite ($\langle a_0, a_1, a_2, \ldots\rangle$).

We saw two specific examples of sequences last time, arithmetic sequences (of the form $\langle c, c+k, c+2k, \ldots\rangle$) and geometric sequences (of the form $\langle c, cr, cr^2, \ldots\rangle$).

Further recall the Method of Mathematical Induction (MMI):

Basis: Show that $P(0)$ holds. Induction: Assume $P(k)$ holds, and show that $P(k+1)$ holds. Then, MMI tells us that $P(n)$ holds for all $n \in \mathbb{N}$.

As we saw last time, MMI is a very powerful tool!

0) a) Re-write the explicit and recursive definitions for the arithmetic and geometric sequences.

b) Find (with proof) the recursive and explicit definitions for the sequence $\langle 0, 1, 3, 6, 10, 15, \ldots\rangle$.

c) Find (with proof) the recursive and explicit definitions for the sequence $\langle 1, 2, 5, 10, 17, 26, \ldots\rangle$.

d) Find (with proof) the recursive and explicit definitions for the sequence $\langle 8, 27, 64, 125, 216, \ldots\rangle$.

Finite Series

A *series* is the sum of the terms of a sequence. Suppose $\langle a_n | n \in \mathbb{N}\rangle$ is a sequence. We call the finite series S_k

$$S_k = \sum_{n=0}^{k} a_n = a_0 + a_1 + \cdots + a_k$$

the partial sums (associated with the sequence $\langle a_n\rangle$).

1) Find (with proof) S_k for the following sequences:

a) The sequence with $a_n = 1$ for all $n \in \mathbb{N}$.

b) The sequence with $a_n = n$ for all $n \in \mathbb{N}$..

c) The sequence with $a_n = 2n+1$ for all $n \in \mathbb{N}$..

2) Find (with proof) S_k for an arithmetic sequence.

3) Prove that

$$S_k = c \cdot \frac{1 - r^{k+1}}{1 - r}$$

for a geometric sequence.

Limits (Informal)

Suppose that $\langle a_n | n \in \mathbb{N} \rangle$ is a sequence. We say the *limit* as n goes to infinity of $\langle a_n \rangle$ is equal to L if a_n gets closer and closer to L as n gets larger. In symbols we write

$$\lim_{n \to \infty} a_n = L.$$

If a sequence has such a limit, we call it *convergent*. Otherwise, it is *divergent*.

4) For the following sequences, write out recursive and explicit formulas for each of the following. Then try to guess the limit of the sequence or say that it diverges.

a) $\langle 1, 2, 3, 4, \ldots \rangle$

b) $\langle 1, 1, 1, 1, \ldots \rangle$

c) $\langle 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \rangle$

d) $\langle 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots \rangle$

e) $\langle 2, -2, 2, -2, \ldots \rangle$

Infinite Series

Suppose that $\langle a_n | n \in \mathbb{N} \rangle$ is a sequence. We can then form the sequence of partial sums $\langle S_k | k \in \mathbb{N} \rangle$. If the sequence $\langle S_k \rangle$ has a limit L say the infinite series $\sum_{n=0}^{\infty} a_n$ *converges* to L and write

$$L = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots$$

Otherwise we say the series is *divergent*.

5) For each of the following series, say if you think the series is convergent or divergent. If it is convergent, try to guess the limit.

a) $\sum_{n=0}^{\infty} 1$

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$

c) $\sum_{n=0}^{\infty} \frac{1}{n!}$

d) $\sum_{n=0}^{\infty} \frac{1}{n^2}$

e) $\sum_{n=0}^{\infty} \frac{1}{n}$

f) $\sum_{n=0}^{\infty} (-1)^n$

Limits (Formal)

Let $\langle a_n | n \in \mathbb{N} \rangle$ be a sequence. Then $\lim_{n\to\infty} a_n = L$ if and only if

$$\text{For any } \varepsilon > 0, \text{ there is an } N, \text{ such that for any } n \geq N : \ |a_n - L| < \varepsilon.$$

Think of this as a game: I say how close I want to get to the limit (i.e. ε), and you tell me how far I need to go in the sequence (i.e. N).

6) Prove the following using the formal definition of limits:

a) $\lim_{n\to\infty} a_n = 0$, where $a_n = 1/n$.

b) $\lim_{n\to\infty} (1 - 2^{-n}) = 1$.

c) $\lim_{n\to\infty} \left(\frac{n}{n+1} \right) = 1$.

7)* Suppose $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Prove that

$$\lim_{n\to\infty} (a_n + b_n) = L + M.$$

Hint: Recall that $|A + B| \leq |A| + |B|$.

8) a) Write out a formal definition for

$$\sum_{n=0}^{\infty} a_n = L.$$

b) Using this formal definition, prove that if $|r| < 1$ then

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}.$$