LAMC Beginners' Circle **June 11, 2014**

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Warm-up

Problem 1 *Simplify the following fraction.*

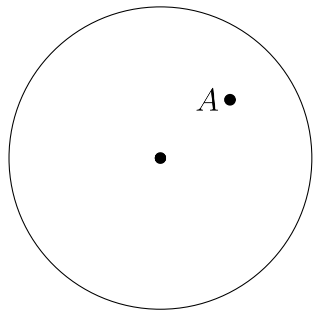
$$\frac{\frac{1}{2}}{\frac{1}{3}-\frac{1}{4}} =$$

Problem 2 *Alice and Bob use some floor scales to find the weight of their school bags. When Alice put her bag on the scales, the scales showed 3 lbs. When Bob put his bag on the scales, the scales showed 2 lbs. When the children put their bags on the scales together, the scales showed 6 lbs!*

- *How can this happen?*

- *How much do the bags really weigh?*

Problem 3 *Point A is marked inside a circle as on the picture below.*



- *Cut the circle into three parts such that by rearranging the parts one can get a circle centred at A.*

- *Is it possible to cut the circle into two parts such that by rearranging the parts one can get a circle centred at A? Why or why not?*

Problem 4 *Find thee integers x, y, and z that solve the following equation.*

$$28x + 30y + 31z = 365$$

$x =$ $\qquad$ $y =$ $\qquad$ $z =$

The 15 puzzle, permutations and taxicab distance

The ultimate goal of the mini-course we begin now is to figure out which configurations of the *15 puzzle* are solvable.

The puzzle consists of a 4×4 frame randomly filled with 15 squares numbered one through fifteen. The objective is to slide the squares in the proper order, left to right, starting with the top row as on the picture below.

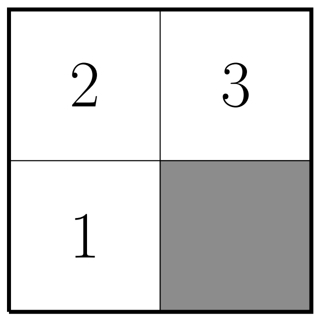
15 puzzle

There are two mathematical staples of the game's theory, *permutations* and the *taxicab distance* (also known as the l_1 or L_1 metric). Both have a wide variety of applications in math, physics, and engineering.

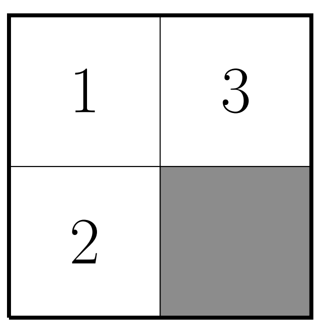
The original 15 puzzle may be a bit too hard to tackle right away. Let us practice playing its more elementary analogue, the 3 puzzle first.

To solve the following 3 puzzle problems, please cut out the gray box and the white squares numbered one through three from the last page of this handout.

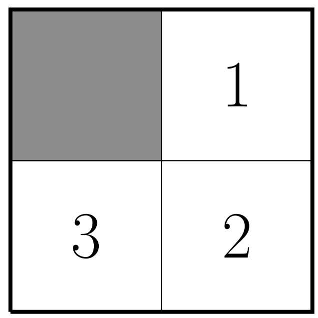
Problem 5 *Solve the following configuration of the 3 puzzle.*



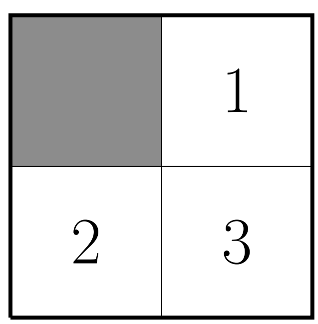
Problem 6 *Solve the following configuration of the 3 puzzle.*



Problem 7 *Solve the following configuration of the 3 puzzle.*



Problem 8 *Solve the following configuration of the 3 puzzle.*

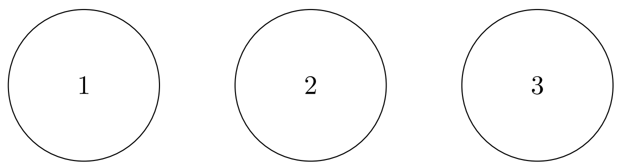


Some of the above configurations of the 3 puzzle were solvable, some weren't.

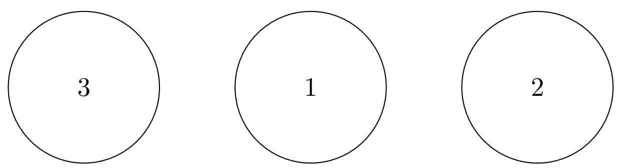
Question 1 *Is there an easy way to tell a solvable configuration from an unsolvable one?*

Permutations

Consider a set of marbles numbered 1 through n. Originally the marbles are lined up in the order given by their numbers. The following picture shows an example with $n = 3$.



Then the marbles are shuffled in a different order.



A *permutation* is the operation of shuffling the marbles. The permutation shown above is written down as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

To understand how permutations work, it is important to learn ... talking to them! Here is a dialogue that explains what the above permutation does.

– Ms. Permutation, what object do you put in the first position?
– The one that used to be in the third.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & & \end{pmatrix}$$

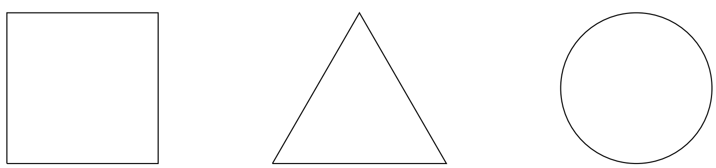
– Ms. Permutation, what object do you put in the second position?
– The one that used to be in the first.

$$\begin{pmatrix} 1 & 2 & 3 \\ & 1 & \end{pmatrix}$$

– Ms. Permutation, what object do you put in the third position?
– The one that used to be in the second.

$$\begin{pmatrix} 1 & 2 & 3 \\ & & 2 \end{pmatrix}$$

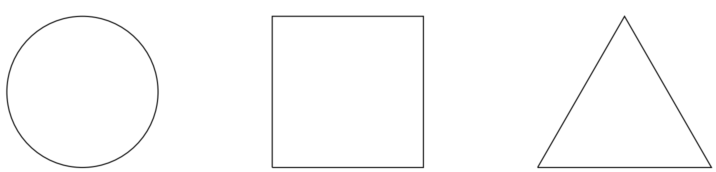
Instead of the numbered marbles, we can shuffle elements of any set. For example, let us use the following geometric figures.



The permutation

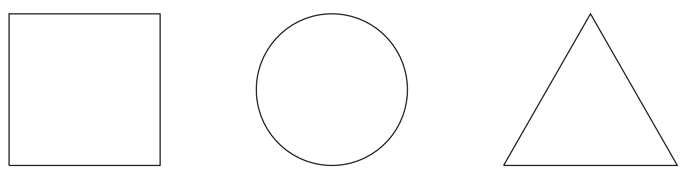
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

considered above shuffles the figures in the following order.

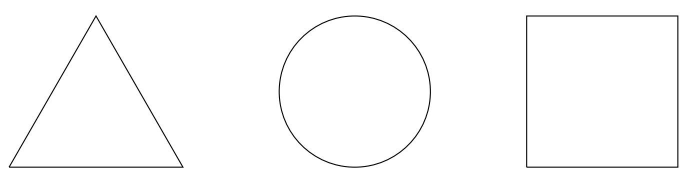


According to the dialogue with Ms. Permutation, the object that used to be in the third position, a circle, gets moved to the first. The object that used to originally be in the first position, a square, is shuffled to the second. Finally, the object that used to be in the second position, a triangle, is moved to the third.

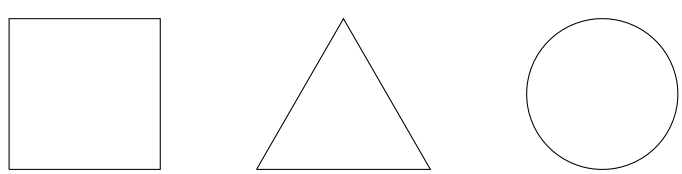
Problem 9 *For the original order of figures on page 7, write down the permutations that correspond to the following pictures.*



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that the last permutation does not shuffle anything at all. Permutations of this kind are typically denoted as e and called *trivial*. A trivial permutation is still a permutation, and an important one!

Problem 10 *Write down the trivial permutation for $n = 5$.*

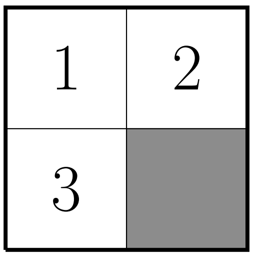
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

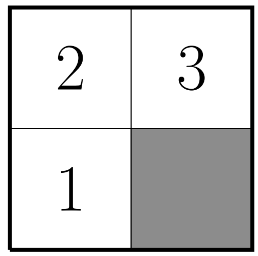
Problem 11 *For the original order of figures introduced on page 7, draw the figures in the orders prescribed by the permutations below. Use the space to the right of a permutation to draw the corresponding picture.*

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

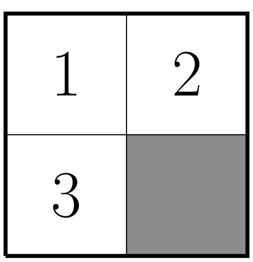
Problem 12 *On the left-hand side of the following picture is the original, and the winning, configuration of the 3 puzzle. The configuration from Problem 5 is on the right-hand side. Considering the empty square as the fourth object, write down the permutation that shuffles the first configuration into the second.*

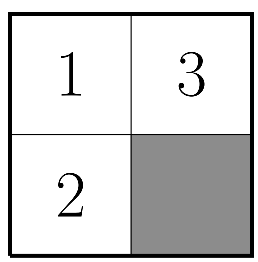




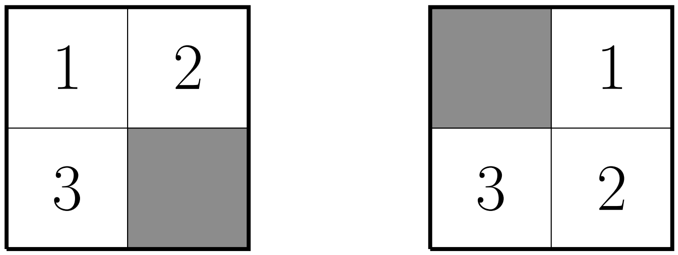
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Do the same for the configurations from Problems 6 and 7.



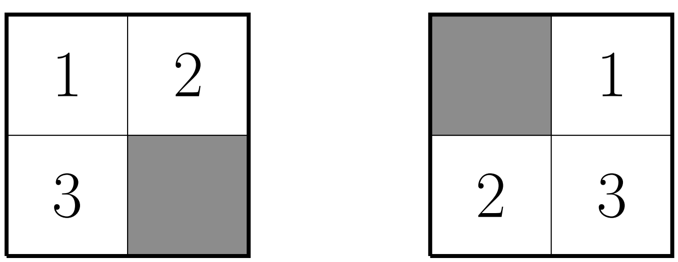


$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 13 *Write down the permutation that shuffles the configuration from Problem 8, on the right-hand side of the picture below, into the winning configuration on the left-hand side of the picture. Hint: ask Ms. Permutation what to do!*



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Multiplication of permutations

It is possible to combine, or *multiply*, permutations. For example, let us apply the permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

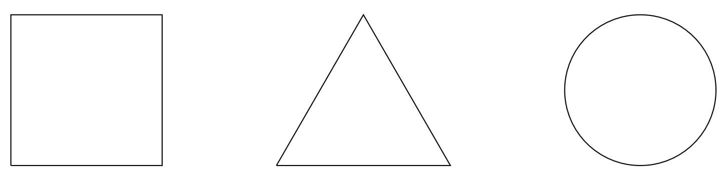
to the marbles already shuffled by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The permutation δ switches the first and second elements, so

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

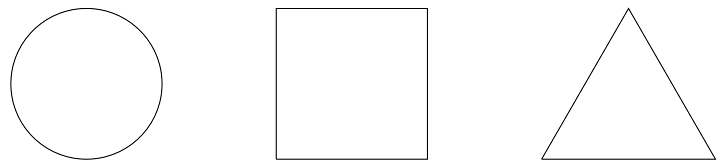
Let us take another look at the above computation using the figures introduced on page 7. Originally, the set of the figures is ordered as follows.



The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

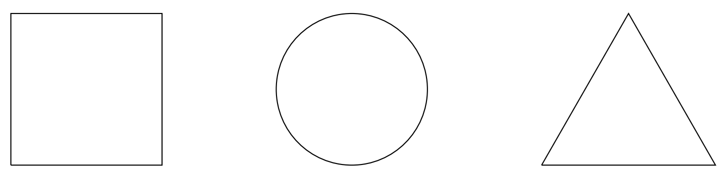
produces the picture below.



The permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

applied to the latter configuration gives us the following.



Comparing the last picture to the original gives us the answer.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note that in the product $\delta \circ \sigma$ of permutations, it is the one on the right, σ, that acts first on the set it permutes!

Problem 14 *Find the permutation $\sigma \circ \delta$. If needed, use the pictorial representation as above.*

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Is $\sigma \circ \delta = \delta \circ \sigma$*?*

An operation $\circ$ acting on two elements α and β is called *commutative* if $\alpha \circ \beta = \beta \circ \alpha$ for all the possible inputs α and β. For example, addition and multiplication of numbers are commutative, $\alpha + \beta = \beta + \alpha$ and $\alpha \times \beta = \beta \times \alpha$ for any two numbers α and β.

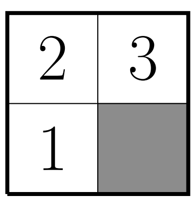
Question 2 *Why is it always true that* $\alpha \times \beta = \beta \times \alpha$ *for any two (real) numbers* α *and* β*?*

Multiplication of permutations is not commutative in general, as shown in Problem 14. However, some particular permutations can commute.

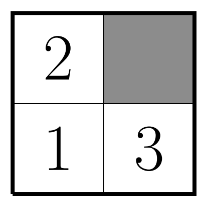
Problem 15 *Find two non-trivial permutations of four elements that do commute.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

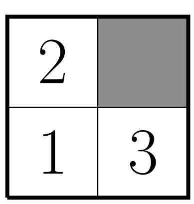
Let us take another look at the solution of the 3 puzzle from Problem 5. The first move switches the square numbered 3 in the second position and the empty square in the fourth one.



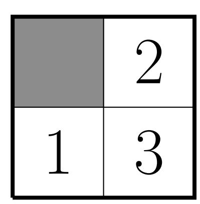
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$



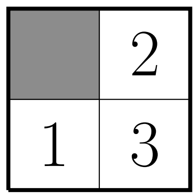
The next move switches the square numbered 2 in the first position and the empty square in the second position.



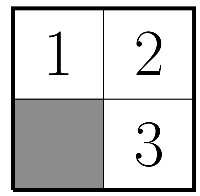
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$



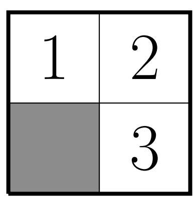
The next move switches the square numbered 1 in the third position and the empty square in the first position.



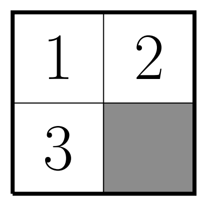
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$



The last and final move switches the square numbered 3 in the fourth position and the empty square in the third position.



$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$



This way, the winning configuration is obtained from the original one by means of applying the following product of permutations.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Problem 16 *Find the above product explicitly. Hint: there is a way much shorter than multiplying the above four permutations directly!*

$$\textit{the product} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 17 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

If you need to use a pictorial representation as a tool, take the one on page 7 and add a diamond $\diamondsuit$ as the fourth figure.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 18 *Find the product $\sigma \circ \delta$ of the permutations*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \textit{and} \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

from Problem 17. If needed, use a pictorial representation.

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Do the permutations δ *and* σ *commute?*

For a positive integer n, the following defines the number $n!$ (reads n factorial).

Definition 1 $$n! = 1 \times 2 \times \ldots \times (n-1) \times n$$

For example, $3! = 1 \times 2 \times 3 = 6$.

Problem 19 *Find the number of all different permutations of three elements.*

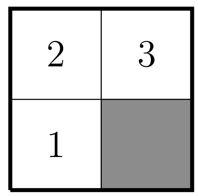
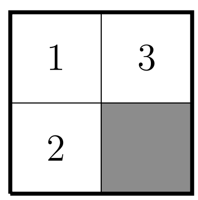
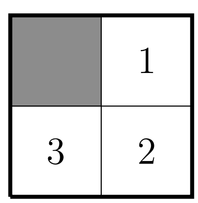
Problem 20 *Find the following number.*

$$4! =$$

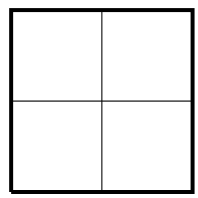
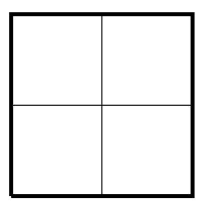
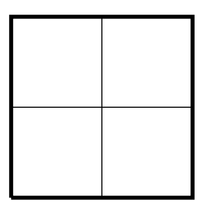
Problem 21 *Find the number of all different permutations of four elements.*

Homework

In Problem 12, we have found the permutations corresponding to the 3 puzzle configurations considered in Problems 5, 6, and 7. We know that the configurations from Problems 5 and 7 are solvable while the configuration from Problem 6 is not. Let us organize all this information in the following table.

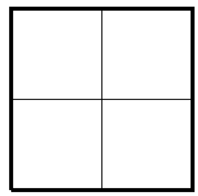
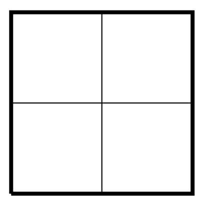
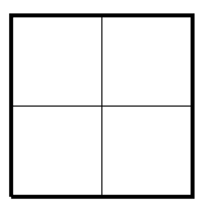
permutation	3 puzzle conf.	solvable
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$		Y
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$		N
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$		Y

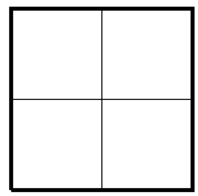
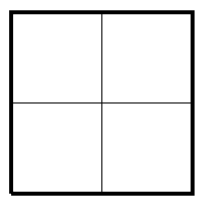
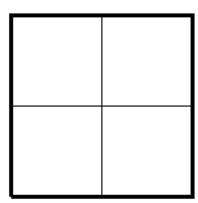
Problem 22 *Complete the above table, continued below. List the remaining 21 different permutations of four objects. Draw the corresponding 3 puzzle configurations. For each configuration, decide whether it is solvable.*

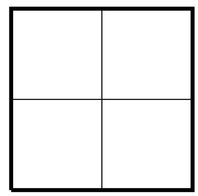
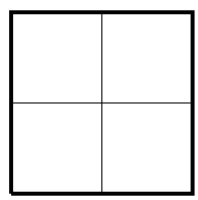
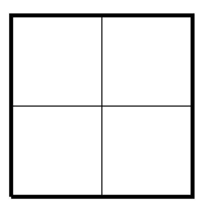
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

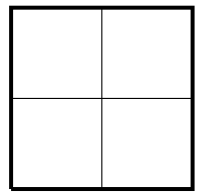
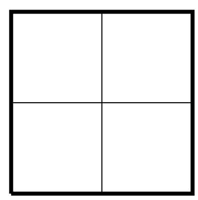
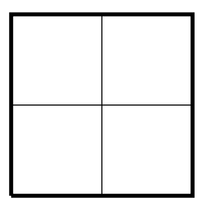
permutation	*3 puzzle conf.*	*solvable*
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

permutation	*3 puzzle conf.*	*solvable*
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

permutation	*3 puzzle conf.*	*solvable*
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

permutation	*3 puzzle conf.*	*solvable*
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

permutation	*3 puzzle conf.*	*solvable*
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

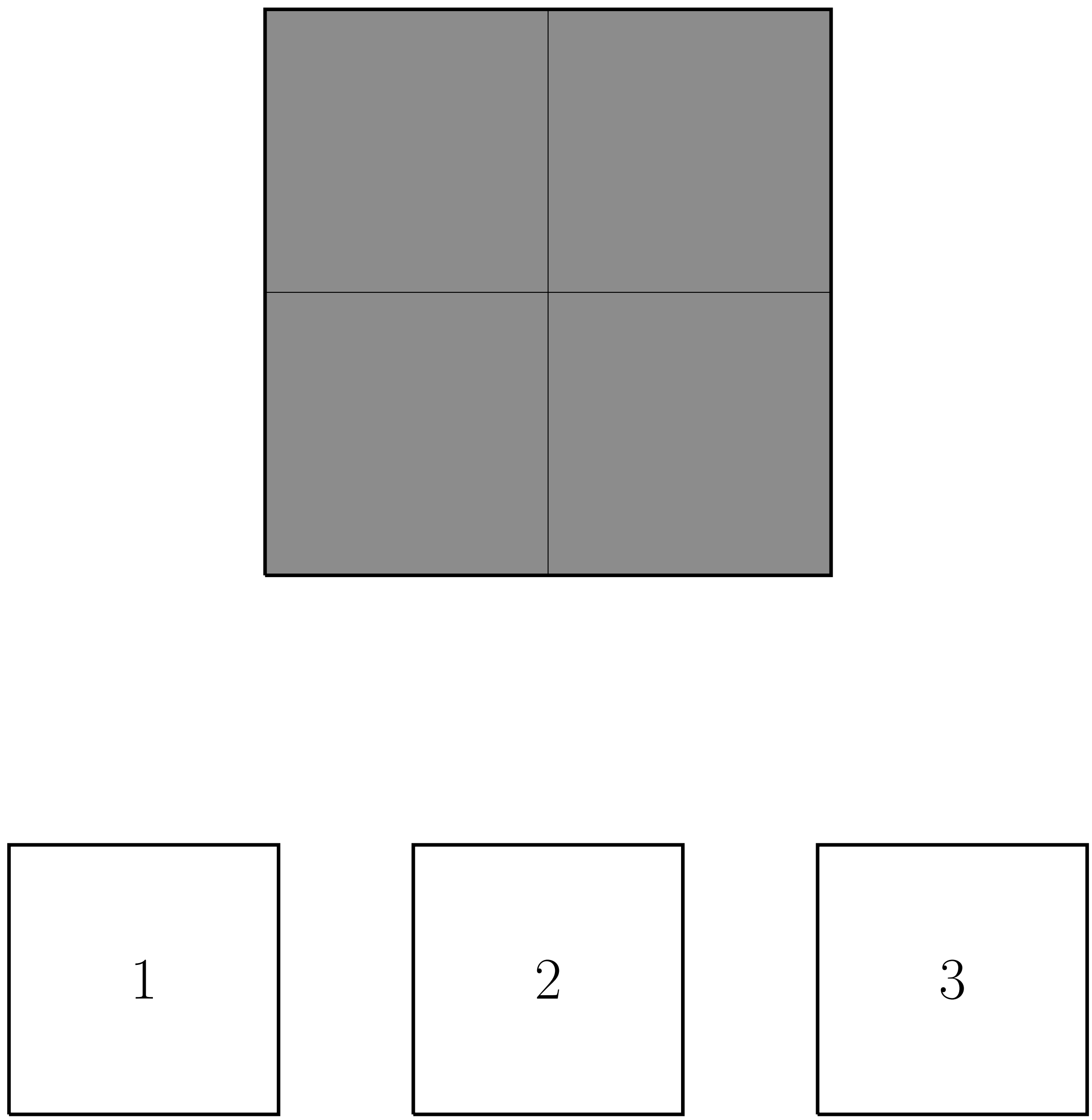
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		
$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$		

How many solvable configurations of the 3 puzzle are there? How many unsolvable configurations of the puzzle are there?

Problem 23 *Take a good look at the complete table. Try to figure out what makes a configuration of the 3 puzzle unsolvable.*

Problem 24 *(Oldaque P. de Freitas Puzzle)*

Two ladies are sitting in a street café, talking about their children. One lady says that she has three daughters. The product of the girls' ages equals 36 and the sum of their ages is the same as the number of the house across the street. The second lady replies that this information is not enough to figure out the age of each child. The first lady agrees and adds that her oldest daughter has beautiful blue eyes. Then the second lady solves the puzzle. Please do the same. Do not use Internet to look for a solution!


1
2
3