SYMBOLIC LOGIC

(PART TWO)

LAMC INTERMEDIATE - 6/1/14

Warm Up

(1) Suppose we have a triangle ABC. Let M_A be the midpoint of side BC, M_B be the midpoint of side AC, and M_C be the midpoint of side AB. Express the following in terms of the vectors $\vec{v} = \overrightarrow{AB}$ and $\vec{w} = \overrightarrow{AC}$. ***(DRAW A PICTURE!!)***

(a) $\overrightarrow{AM_A}$

(b) $\overrightarrow{BM_B}$

(c) $\overrightarrow{CM_C}$

(2) The notation from the previous problem continues here. Compute each of the following in terms of $\vec{v}$ and $\vec{w}$:

(a) $\frac{2}{3}\overrightarrow{AM_A}$

(b) $\vec{v} + \frac{2}{3}\overrightarrow{BM_B}$

(c) $\vec{w} + \frac{2}{3}\overrightarrow{CM_C}$

(3) Prove that all three medians of a triangle intersect at a single point.

(4) (Midpoint theorem) Let ABC be a triangle. Let M be the midpoint of AB and N be the midpoint of AC. Show that the length of MN is half the length of BC. *(Hint: Express each side of the triangle with vectors and show that* $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$*!)*

NEGATION AND DE MORGAN'S LAWS

(1) Show that $\neg(P \vee Q)$ is logically equivalent to $\neg P \wedge \neg Q$.

(2) Show that $\neg(P \wedge Q)$ is logically equivalent to $\neg P \vee \neg Q$.

(3) Prove or disprove: $\neg(P \to Q)$ is logically equivalent to $\neg P \to \neg Q$.

(4) Prove or disprove: $\neg(P \to Q)$ is logically equivalent to $P \wedge \neg Q$.

Satisfiability

A formula (or collection of formulas) is called **satisfiable** if there is a **model** in which the formula (or collection of formulas) is true. For us, a **model** is simply a truth assignment to all propositional variables. For example, if $P, Q,$ and R are our propositional variables a model could be

P	Q	R
T	**F**	**T**

In this model $P \wedge Q$ is false, but $P \wedge R$ is true. A formula is thus satisfiable if we can come up with a model in which it is true.

(1) Let $P, Q,$ and R be propositional variables. Determine whether or not the following are satisfiable. If you think a formula is satisfiable, provide a model which satisfies it.

(a) $\neg Q \vee Q$

(b) $\neg P \wedge P$

(c) $\neg(P \to Q) \wedge R$

(d) $(P \to Q) \leftrightarrow (\neg Q \to P)$

(e) $(P \wedge \neg P) \vee \neg Q$

(f) $\{\neg(P \to Q) \wedge \neg R,\ \ (P \wedge \neg R) \vee \neg Q\}$

(g) $\{\neg(P \to Q),\ \ \neg R,\ \ (P \vee Q) \leftrightarrow R\}$

(h) $\{(P \wedge \neg R) \vee \neg Q,\ \ (P \vee Q) \leftrightarrow R\}$

(i) $\{P \to Q,\ \ \neg Q \to \neg P,\ \ \neg P \vee Q\}$

The Compactness Theorem. A set of formulas has a model if and only if every finite subset has a model. (*For the remainder of the handout, we will assume this theorem is true)*

(1) Prove the forward direction of the Compactness Theorem. That is, prove that if a set of formulas has a model, then every finite subset has a model.

(2) **(Non-Standard Analysis)** In this problem, we will construct a model of the real numbers which contains infinitesimally small numbers. (Such numbers are the basis for how Leibniz perceived of calculus)

(a) Let Σ be a set of formulas which defines the real numbers. Let ϵ be a new symbol. Let n be arbitrary and consider the proposition $\epsilon < \frac{1}{n}$. Is this proposition satisfiable?

(b) Show that the propositions $\epsilon < 1, \cdots, \epsilon < \frac{1}{n}$ are satisfiable.

(c) Use the compactness theorem to conclude that the collection $\left\{\epsilon < \frac{1}{n} : n \in \mathbb{N}\right\}$ is satisfiable.

(d) Prove that there is a model of the real numbers with infinitely large elements.

Tautologies and Tautological Implication

We say that a set of formulas, Σ, **tautologically implies** a formula θ, written $\Sigma \models \theta$, if every model of Σ is also a model of θ. We say that a formula θ is a **tautology** if every set of formulas tautologically implies θ.

(1) Show that
$$\{P, P \to \neg Q, Q \leftrightarrow R\} \models \neg R.$$

(2) Show that
$$\{P, \neg P\} \models ((Q \to P) \wedge (R \vee (Q \leftrightarrow \neg P))) \to \neg(Q \leftrightarrow R)$$

(3) Show that
$$\{P \wedge (Q \to R), \neg R\} \models \neg Q$$

(4) Come up with a tautology.

(5) Of the following formulas, which imply which? (That is, does $\{\theta_1\} \models \theta_2$? etc.)

(a) $\theta_1 = (P \to Q)$

(b) $\theta_2 = \neg((P \to Q) \to \neg(Q \to P))$

(c) $\theta_3 = (\neg P \vee Q) \wedge (P \vee \neg Q)$

(6) Prove the following:

(a) **(Deduction Theorem)** $\Sigma \cup \{\theta\} \models \varphi$ if and only if $\Sigma \models \theta \to \varphi$.

(b) If $\Sigma \models \varphi$ or $\Sigma \models \theta$, then $\Sigma \models \varphi \vee \theta$.

(c) Is the converse of (b) true? That is, if $\Sigma \models \varphi \vee \theta$ does $\Sigma \models \varphi$ or $\Sigma \models \theta$? Prove or provide a counterexample.

Challenge Problems

(1) Suppose that Σ is a set of propositions. Suppose further that for every model M there is some formula $\sigma \in \Sigma$ such that $M \models \sigma$. Prove that there are formulas $\sigma_1, \cdots, \sigma_n \in \Sigma$ such that

$$\sigma_1 \vee \sigma_2 \vee \cdots \vee \sigma_n$$

is a tautology.

(2) Prove the following theorem: *If $\Sigma \models p$ then there is a finite subset Σ_0 of Σ such that $\Sigma_0 \models p$.*