Modular Arithmetic, Part II

May 4, 2014

Modular arithmetic. Last week, we introduced modular congruencies. Two whole numbers a and b are said to be *congruent modulo* n, often written $a \equiv b \pmod{n}$, if they give the same remainders when divided by n.

Warm Up

Draw a picture showing mod 8 arithmetic on a circle (similar to mod 12 arithmetic on a clock).

(1) Reduce the numbers in modular arithmetic.

(a) $17 \equiv \quad \pmod 5$

(b) $29 \equiv \quad \pmod{10}$

(c) $433551 \equiv \quad \pmod 2$

(d) $91 \equiv \quad \pmod{13}$

(e) $-1 \equiv \quad \pmod 5$

(f) $-10 \equiv \quad \pmod 6$

(g) $-1 \equiv \quad \pmod n$

(2) Fill in the addition and multiplication tables:

+	EVEN	ODD
EVEN		
ODD		

$\times$	EVEN	ODD
EVEN		
ODD		

(3) Fill in the addition and multiplication tables below in the case of modulus $n = 2$:

+	0	1
0		
1		

$\times$	0	1
0		
1		

(a) Any even number can be represented by $2 \times n$. Reduce this number in mod 2 arithmetic:

$$2 \times n \equiv \qquad \pmod 2$$

(b) Any odd number can be represented by $2 \times n + 1$. Reduce this number in mod 2 arithmetic:

$$2 \times n + 1 \equiv \qquad \pmod 2$$

(4) There are 931 students in Franklin Elementary School. During an assembly, all of the students gather together and form a grid. The figure below shows what the grid looks like. The number of students in each row is even, and the number of students in each column is even. Did all of the students attend the assembly? Explain your answer.

(5) Lucy went to the store to buy milk, bread, eggs, and pencils. Milk costs $\$4.42$, bread costs $\$1.64$, pencils cost $\$0.72$, and eggs cost $\$4.98$. Her total bill ended up being $\$31.63$. Is this possible? Explain why or why not.

(6) Fill in the addition and multiplication tables in the cases of modulus $n = 4$, $n = 5$, and $n = 6$.

(a) Modulo 4:

$+$	0	1	2	3
0				
1				
2				
3				

$\times$	0	1	2	3
0				
1				
2				
3				

(b) Modulo 5:

$+$	0	1	2	3	4
0					
1					
2					
3					
4					

$\times$	0	1	2	3	4
0					
1					
2					
3					
4					

(c) Modulo 6:

$+$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

$\times$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

(d) A non-zero number k is called a *zero divisor* in mod n arithmetic if there is another number l such that $k \times l$ is equivalent to zero in arithmetic mod n.

List all the zero divisors for

- $n = 4$
- $n = 5$
- $n = 6$

(e) Explain the name *zero divisor*.

(f) What is special about the case $n = 5$?

(g) Can you find another n that has this property? Explain.

(7) Use the addition tables you filled out in the previous problems to solve the following subtraction problems.

$$3 - 2 \equiv \qquad \pmod 4$$

$$4 - 3 \equiv \qquad \pmod 5$$

$$4 - 1 \equiv \qquad \pmod 5$$

$$5 - 2 \equiv \qquad \pmod 6$$

(8) Use the multiplication tables you filled out to solve the following division problems

$$3 \div 1 \equiv \qquad \pmod 4$$

$$3 \div 2 \equiv \qquad \pmod 5$$

$$4 \div 3 \equiv \qquad \pmod 5$$

$$2 \div 3 \equiv \qquad \pmod 5$$

$$5 \div 1 \equiv \qquad \pmod 6$$

Properties of Congruences:

- $a \equiv a \pmod{d}$
- $a \equiv b \pmod{d}$ implies $b \equiv a \pmod{d}$
- If $a \equiv b \pmod{d}$ and $b \equiv c \pmod{d}$, then $a \equiv c \pmod{d}$.
- If $a \equiv a' \pmod{d}$ and $b \equiv b' \pmod{d}$, then
  - $a \pm b \equiv a' \pm b' \pmod{d}$
  - $ab \equiv a'b' \pmod{d}$

(9) Reduce the following expressions in modular arithmetic:

$700 + 14 + 23 + 778 \equiv \qquad \pmod{7}$

$89 - 23 - 12 + 9 \equiv \qquad \pmod{11}$

$8 \times 4 \times 7 \times 18 \equiv \qquad \pmod{3}$

$23 \times 51 \times 6 \times 17 \equiv \qquad \pmod{5}$

$3 \times 7 \times 11 \times 17 \times 23 \times 29 \times 113 \equiv \qquad \pmod{12}$

(10) In decimal notation, we write numbers as sums of powers of 10. For instance, 7512 is equal to the following sum:

$$7512 = \mathbf{7} \times 1000 + \mathbf{5} \times 100 + \mathbf{1} \times 10 + \mathbf{2} \times 1$$

We will use this to explain a test for dividing by 3.

(a) Reduce the following numbers in mod 3 arithmetic:

$1 \equiv \qquad \pmod 3$

$10 \equiv \qquad \pmod 3$

$100 \equiv \qquad \pmod 3$

$1000 \equiv \qquad \pmod 3$

$1\underbrace{0\ldots0}_{n} = \underbrace{10 \times 10 \times \cdots \times 10}_{n} \equiv \qquad \pmod 3$

(b) Complete the following statement:

A number is divisible by $3 \Leftrightarrow$ it is equivalent to $\square$ modulo 3.

(c) Now let's use this to determine if 7512 is divisible by 3. In the expansion of 7512 replace 10, 100 and 1000 by a number it is equivalent to in mod 3 arithmetic:

$$\begin{aligned} \mathbf{7} \times 1000 + \mathbf{5} \times 100 + \mathbf{1} \times 10 + \mathbf{2} \times 1 &\equiv \mathbf{7} \times 1 + \mathbf{5} \times 1 + \mathbf{1} \times 1 + \mathbf{2} \times 1 \\ &= 7 + 5 + 1 + 2 \equiv \qquad \text{mod } 3. \end{aligned}$$

(d) Is 7512 divisible by 3? Explain. Than check your answer using long division.

(e) Formulate a rule you can use to determine if a number is divisible by 3 (without performing a division)?

(11) State whether the following numbers are divisible by 3.

(a) 1362

(b) 9647

(c) 5343

(d) 189