Proofs Without Words

LA Math Circle | Advanced Group

May 4, 2014

DIRECTIONS. For your solutions to these problems, you may only rely on basic algebra (and facts like $x^2 \geq 0$ for all x) or on problems that you have already solved. You may not, for example, rely on a theorem you found in a book, or something like the quadratic formula (unless you prove it in the course of your solution).

1. Give your own proof without words of the following theorem.
   Theorem. The sum of the first n whole numbers is equal to half the product of n and $n+1$. In other words,

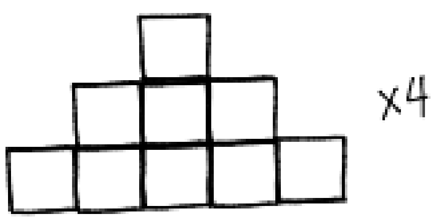
$$1 + 2 + 3 + ... + n = \frac{1}{2}n(n+1).$$

   (Hint: The sequence of numbers given by $\frac{1}{2}n(n+1)$, for $n = 1, 2, 3, ...$ is the triangular numbers)

2. Prove that the sum of the first n odd numbers is equal to n^2. First write this down as a mathematical equation, then identify a puzzle piece that might help you prove it, then prove it.
Draw separate pictures for the cases $n = 1,\ n = 2,\ n = 3,\ n = 4$, then explain exactly how to generalize for any number n.

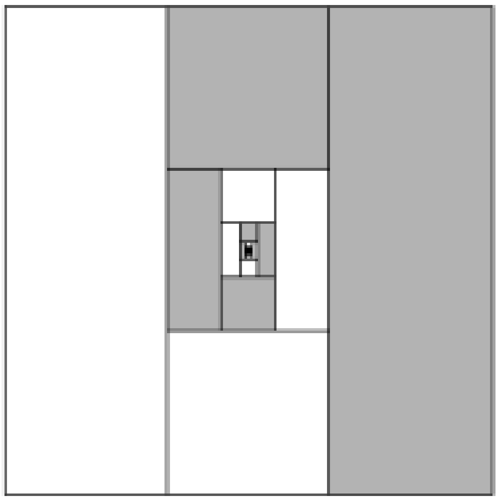
3. Prove the same theorem as you did in Problem 2, except this time, use the following puzzle pieces.

Figure 0.1: Problem 3.



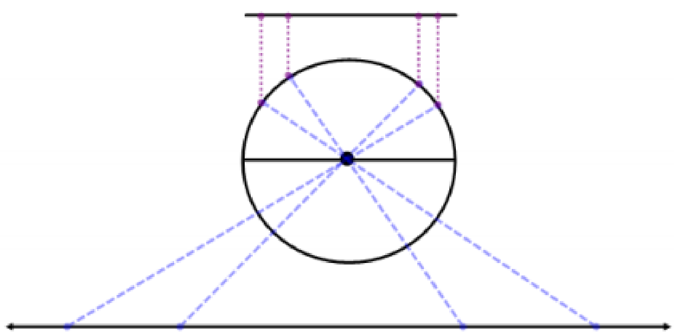
4. Explain this picture, mentioning the theorem it is proving.

Figure 0.2: Problem 4.



5. Think of all the points on the number line. Now think of just the points between -1 and 1. Intuitively, it seems that there should be fewer of them.

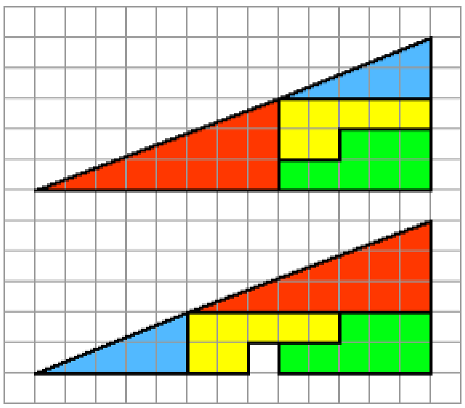
Figure 0.3: Problem 5.



What does the picture tell you?

6. Proofs without words can be misleading if you're not careful. What does the following diagram prove? Can you explain what went wrong?

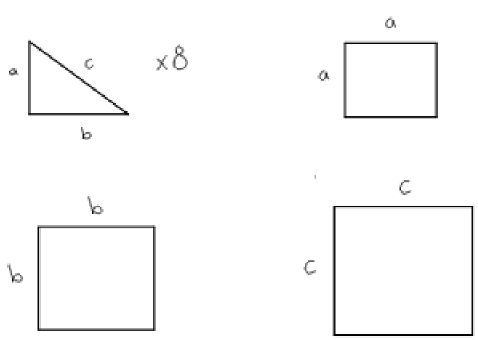
Figure 0.4: Problem 6.



What does the picture tell you?

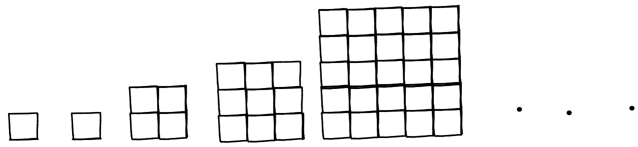
7. Prove the Pythagorean Theorem using the following puzzle pieces:

Figure 0.5: Problem 7.



Hint: You should draw a big square (with side-length $a + b$) in two different ways

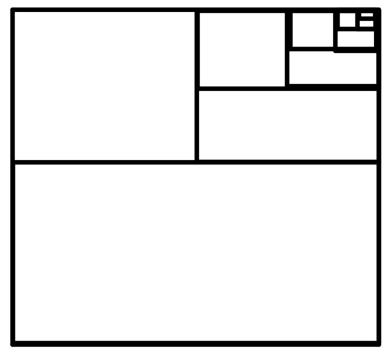
(7) Prove that $F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$ where F_n is the n^{th} Fibonacci number using the following puzzle pieces:



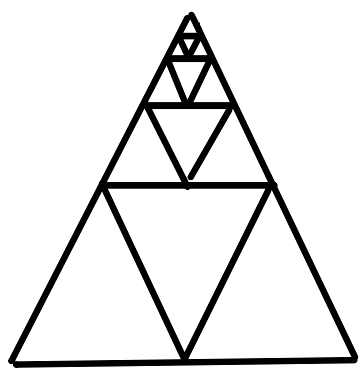
Draw separate pictures for $n = 1, n = 2, n = 3$ and $n = 4$ then explain how exactly to generalize for any number n.

EVALUATE THE INFINITE SUMS!

(1) Evaluate the infinite sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ using the following picture:



(2) Evaluate the infinite sum $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots$ using the following picture:



Picture Proof Jeopardy

In this section, we present a picture proof but don't tell you the theorem it is proving. See if you can figure out what is going on in the picture and provide the theorem it proves.

(1) Hints: Figure out the length and width of this rectangle. Now you can write the area as either length times width or as the sum of the areas of the inside pieces.