VECTORS IN THE CARTESIAN PLANE

LAMC INTERMEDIATE - 4/27/14

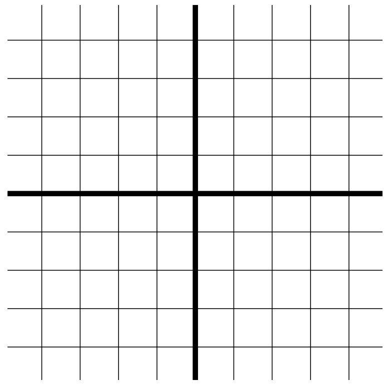
WARM-UP

(Oldaque P. de Freitas Puzzle) Two ladies are sitting in a street cafe, talking about their children. One lady says that she has three daughters. The product of the girls' ages equals 36 and the sum of their ages is the same as the number of the house across the street. The second lady replies that this information is not enough to figure out the age of each child. The first lady agrees and adds that her oldest daughter has beautiful blue eyes. Then the second lady solves the puzzle. Please do the same.

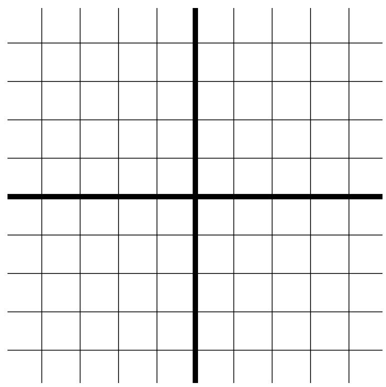
Vectors

(1) A **vector** is a directed arrow between an intial point A and a terminal point B, denoted $\vec{v} = \overrightarrow{AB}$. Draw the vector $\overrightarrow{AB}$ if:

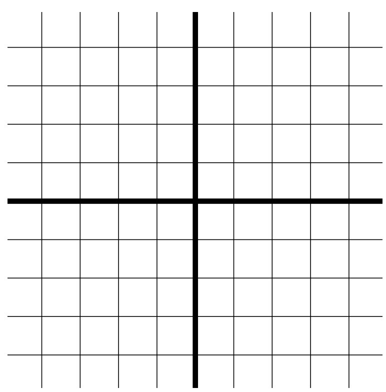
(a) $A = (0, 0)$ and $B = (3, 2)$.



(b) $A = (-3, -2)$ and $B = (1, 4)$.



(c) $A = (-1, 3)$ and $B = (3, 1)$.



(2) Two vectors are considered to be the same if they are translations of each other, in which case we write $\overrightarrow{AB} = \overrightarrow{CD}$. Does $\overrightarrow{AB} = \overrightarrow{CD}$ if:

(a) $A = (0, 0)$, $B = (5, 5)$, $C = (10, 10)$, and $D = (15, 15)$?

(b) $A = (4, 7)$, $B = (4, 3)$, $C = (7, 4)$, and $D = (4, 4)$?

(c) $A = (1, 1)$, $B = (1, 1)$, $C = (10, 10)$, and $D = (10, 10)$?

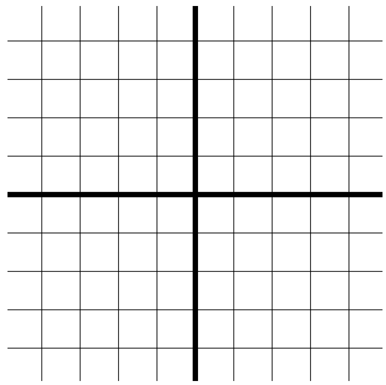
(d) $A = (3, 1)$, $B = (6, 2)$, $C = (11, 9)$, and $D = (8, 8)$?

(e) $A = (2, 5)$, $B = (-3, 4)$, $C = (-31, 435)$ and $D = (-36, 434)$?

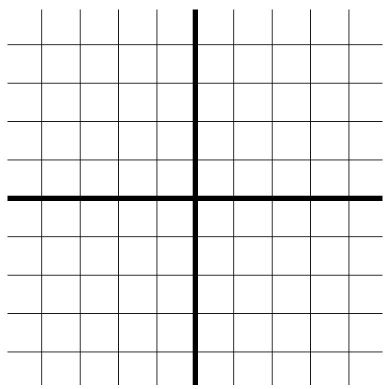
(f) $A = (0, 0)$, $B = (0, 1)$, $C = (0, 0)$ and $D = (1, 0)$?

(3) To add two vectors $\vec{v}$ and $\vec{w}$, translate $\vec{w}$ so that its initial point corresponds with the terminal point of $\vec{v}$. The vector whose initial point is the initial point of $\vec{v}$, and whose terminal point is the terminal point of $\vec{w}$, is the sum and is denoted $\vec{v} + \vec{w}$. Draw $\overrightarrow{AB} + \overrightarrow{CD}$ and explicity state the initial and terminal points if:

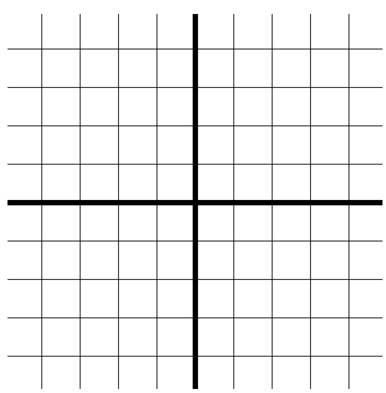
(a) $A = (-4, -3)$, $B = (-2, -3)$, $C = (10, 10)$, and $D = (10, 11)$.



(b) $A = (-3, 0)$, $B = (-3, -4)$, $C = (7, 4)$, and $D = (11, 4)$.



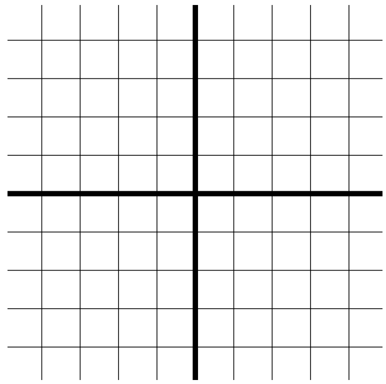
(c) $A = (-4, -4)$, $B = (-1, -1)$, $C = (10, 10)$, and $D = (15, 15)$.



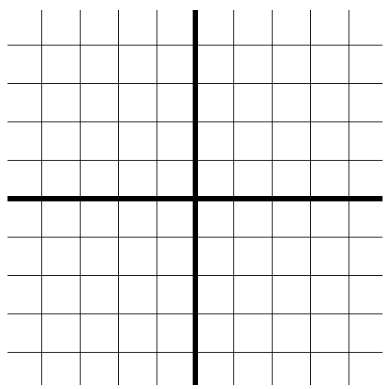
(4) We notice that the "key" feature of a vector is that it has a direction and a magnitude (length). A **scalar** is something which only has a magnitude and no direction. For example, velocity is a vector while speed is a scalar. Please come up with as many physical examples of vectors and scalars as you can.

(5) The last thing we will do is multiply a vector $\vec{v}$ by a scalar λ. The result is the vector $\lambda\vec{v}$ which is the vector which lies on the same line as $\vec{v}$ and has length λ times the length of $\vec{v}$. Draw $\lambda\overrightarrow{AB}$ and determine its initial and terminal points if:

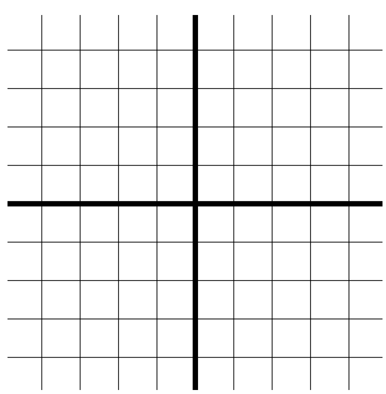
(a) $\lambda = \frac{1}{2}$, $A = (0,0)$ and $B = (2,0)$.



(b) $\lambda = 3$, $A = (0,0)$, and $B = (\frac{4}{3}, \frac{4}{3})$.



(c) $\lambda = \frac{1}{10}$, $A = (0,0)$ and $B = (6,8)$. (*Hint: What is the length of the vector* $\overrightarrow{AB}$*?)*



Cartesian Coordinates and Position Vectors

(1) You already know Cartesian coordinates. They are simply the x, y-coordinates of a point. In this problem we examine what would happen if we moved the origin away from $O = (0, 0)$.

(a) Suppose the origin is moved to the point $(5, 5)$. What are the new coordinates of the point $(0, 0)$?

(b) Suppose the origin is moved to the point $(1, 7)$. What are the new coordinates of the point $(7, 1)$?

(c) Suppose the origin is moved to the point $(2, 6)$. What are the new coordinates of the point $(5, 5)$?

(d) Suppose the origin is moved to the point (x, y). What are the new coordinates of the point (e, f)?

(2) The most important concept is a position vector. The position vector of a point $P = (x, y)$ represents the displacement going from the origin $O = (0, 0)$ to the point P. Numerically the vector is written as $\overrightarrow{OP} = \langle x, y\rangle$.

(a) We can add position vectors by simply adding each component separately, so $\langle x, y\rangle + \langle s, r\rangle = \langle x + s, y + r\rangle$.

(i) Compute $\langle 3, 5\rangle + \langle 2, 6\rangle$.

(ii) Compute $\langle 15, 44\rangle + \langle 14, 93\rangle$.

(iii) Compute $\langle 6, 6\rangle - \langle 4, 2\rangle$.

(iv) Show that this definition of $\overrightarrow{OP} + \overrightarrow{OQ} = \langle x, y\rangle + \langle s, r\rangle$ is the same as the geometric definition we gave earlier for $\overrightarrow{OP} + \overrightarrow{OQ}$. *(Hint: we can let $\overrightarrow{OP}$ have initial point at $(0, 0)$ and terminal point (x, y). Where should you place the initial point of $\overrightarrow{OQ}$?)*

(b) We can also multiply position vectors by a scalar λ by multiplying each component separately, so $\lambda\langle x, y\rangle = \langle \lambda x, \lambda y\rangle$.

(i) Compute $5\langle 3, 2\rangle$.

(ii) Compute $4\langle 1, 6\rangle + 3\langle 2, 9\rangle$.

(iii) Compute $\pi\langle 5, 4\rangle - 2\pi\langle 3, 2\rangle$.

(iv) What is the length of a position vector $\langle x, y\rangle$? What is the length of the position vector $\langle \lambda x, \lambda y\rangle$? Conclude that this definition of scalar multiplication agrees with the geometric definition we gave earlier.

(3) We can also write the vector $\overrightarrow{AB}$ as a pair $\overrightarrow{AB} = \langle x, y\rangle$ by interpreting it as the vector such that $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$, where the addition is coordinate-wise as in the previous problem.

(a) If $\overrightarrow{OA} = \langle 3, 2\rangle$ and $\overrightarrow{OB} = \langle 6, 9\rangle$ what is $\overrightarrow{AB}$?

(b) If $\overrightarrow{OA} = \langle 1, 7\rangle$ and $\overrightarrow{OB} = \langle 7, 1\rangle$ what is $\overrightarrow{BA}$?

(c) If your neighbor wanted to know how to find $\overrightarrow{AB}$ for any $\overrightarrow{OA}$ and $\overrightarrow{OB}$ what would you tell him to do?

(d) Explain why this problem is actually the same as problem 1.

(4) In this problem you will prove that vector addition is commutative.

(a) Prove that

$$\langle x, y\rangle + \langle s, r\rangle = \langle s, r\rangle + \langle x, y\rangle$$

(b) Prove, without using coordinates, that

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

for any vectors $\vec{v}$ and $\vec{w}$.

Geometry Problems with Vectors

(1) Prove that the diagonals of a parallelogram in the Euclidean plane split each other in halves. *(Hint: Express the sides of the paralellogram as vectors)*

(2) Suppose we have a triangle ABC. Express the following in terms of the vectors $\vec{v} = \overrightarrow{AB}$ and $\vec{w} = \overrightarrow{AC}$.

(a) $\overrightarrow{BC}$

(b) $\overrightarrow{CB}$

(c) $\overrightarrow{AM}$, where M is the midpoint of the line BC.

(d) $\overrightarrow{BM}$

(e) $\overrightarrow{CM}$

(3) (Midpoint theorem) Let ABC be a triangle. Let M be the midpoint of AB and N be the midpoint of AC. Show that the length of MN is half the length of BC. ***(Hint: Express each side of the triangle with vectors and show that*** $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{BC}$***!)***

(4) Let ABC be a triangle and let P be a point on BC which divides BC into a ratio of 3 to 2. Describe the vector $\overrightarrow{AP}$ in terms of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. ***(Challenge: Do the same thing, but replace 3 and 2 with m and n)***