LAMC Beginners' Circle Solutions - May 4, 2014

May 9, 2014

Problem 1a: Write down all the inversions of the permutation $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$

Solution: Since an inversion of a permutation is any pair of numbers whose order is changed by the permutation, the inversions are $(4,1), (2,1), (5,1), (3,1), (4,2), (4,3), (5,3)$

Problem 1b: What is the sign of the permutation?

Solution: Since the sign of a permutation is defined as $(-1)^{number\ of\ inversions}$, $sgn(\sigma) = (-1)^7 = -1$

Problem 2: What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

Solution: There are sixteen inversions: $(16,7), (16,8), (16,9), (16,10), (16,11), (16,12), (16,13), (16,14), (16,15), (14,12), (15,12), (13,12), (15,14), (8,7), (9,7), (10,7)$. Since $(-1)^{16} = 1$, the sign of the permutation is 1.

Problem 3: Represent the transposition (41) as a product of adjacent transpositions.

Solution: A product of adjacent transpositions is a permutation that changes the positions of two adjoining elements. Thus, to obtain a permutation that changes the first and fourth elements, move the first element to the second position, the (new) second element to the third position, and so on until the (original) first element is in the fourth position and the (original) fourth element is in the first position. Hence, the product of adjacent transpositions is $(2\,1) \circ (3\,2) \circ (4\,3) \circ (3\,2) \circ (2\,1), since the permutation is applied from right to left.$

Problem 4a: Represent the permutation $\sigma = (4\,2\,5\,3\,1)$ as a product of transpositions.

Solution: A product of transpositions is a series of reorderings that obtain the original permutation one step at a time. Hence $(4\,2\,5\,3\,1)$, which moves the fourth element to the first position, the fifth element to the third position, and the first element to the fifth position, can be written as $(5\,4) \circ (5\,3) \circ (4\,1)$

Problem 4b: Use the formula $sgn(\sigma) = (-1)^t$ where t is the number of transpositions in the product to find the sign of σ. Compare your answer to that of Problem 1.

Solution: $(-1)^3 = -1$ so the sign of σ is -1.

Problem 4c: Did you expect the answers to be the same? Why or why not?

Solution: By Theorem 1 from the handout from April 26th, any adjacent transposition changes the number of inverstions in a permutation by one. Then, because the trivial permutation (the starting point) has zero inversions and three adjacent transpositions represent the permutation, the sign of the permutation is $1 \to -1 \to 1 \quad \to -1$.
Another way of thinking: since the permutation itself hasn't changed, it is logical that the sign of the permutation hasn't changed.