LAMC Beginners' Circle **May 4, 2014**

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Problem 1 *Write down all the inversions of the permutation* $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$.

What is the sign of the permutation?

$$sgn(\sigma) =$$

Problem 2 *What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)*

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

Problem 3 *Represent the transposition* (41) *as a product of adjacent transpositions.*

$$(41) =$$

Problem 4 *Represent the permutation* $\sigma = \begin{pmatrix} 4 & 2 & 5 & 3 & 1 \end{pmatrix}$ *from Problem 1 as a product of transpositions.*

$$\sigma =$$

Use the formula

$$sgn(\sigma) = (-1)^{\#t} \tag{1}$$

where $\#t$ *is the number of transpositions in the product to find the sign of* σ*. Compare your answer to that of Problem 1.*

$$sgn(\sigma) =$$

Did you expect the answers to be the same? Why or why not?

There are two tools needed to prove that the 15 puzzle configuration suggested by Sam Loyd has no solution.

1	**2**	**3**	**4**
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9	**10**	**11**	**12**
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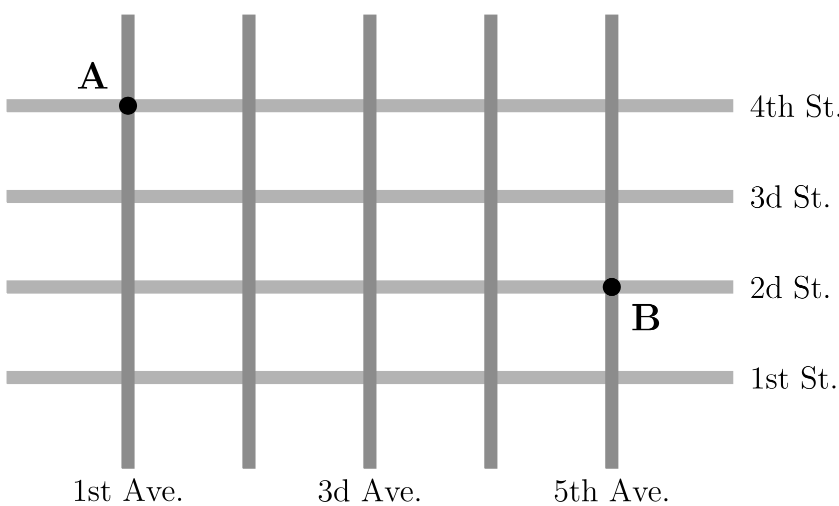
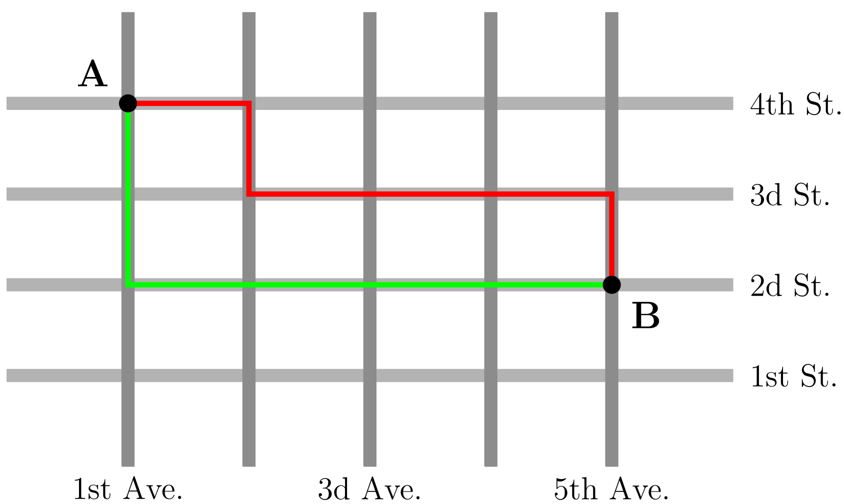
We are already familiar enough with permutations. Now is the time to learn the second tool.

Taxicab geometry

Imagine that you take a taxicab to get from point A to point B in a city with streets and avenues forming a rectangular pattern.



Similar to Euclidean geometry, there exists a shortest path. Unlike Euclidean geometry, the shortest path is not unique. For example, the green and red routes on the picture below are both shortest ways from A to B.



Problem 5 *On the picture above, draw a third shortest path from A to B.*

The point A lies at the intersection of the 1st Ave. and the 4th St. Let us write this fact down as follows.

$$A = (1, 4)$$

B lies at the intersection of the 5th Ave. and the 2nd St.

$$B = (5, 2)$$

Let a be the distance between two neighbouring avenues and let s be the distance between two neighbouring streets. No matter what shortest path the cab driver chooses, he needs to drive

4 blocks East and 2 blocks South.

$$d_{tc}(A, B) = 4a + 2s$$

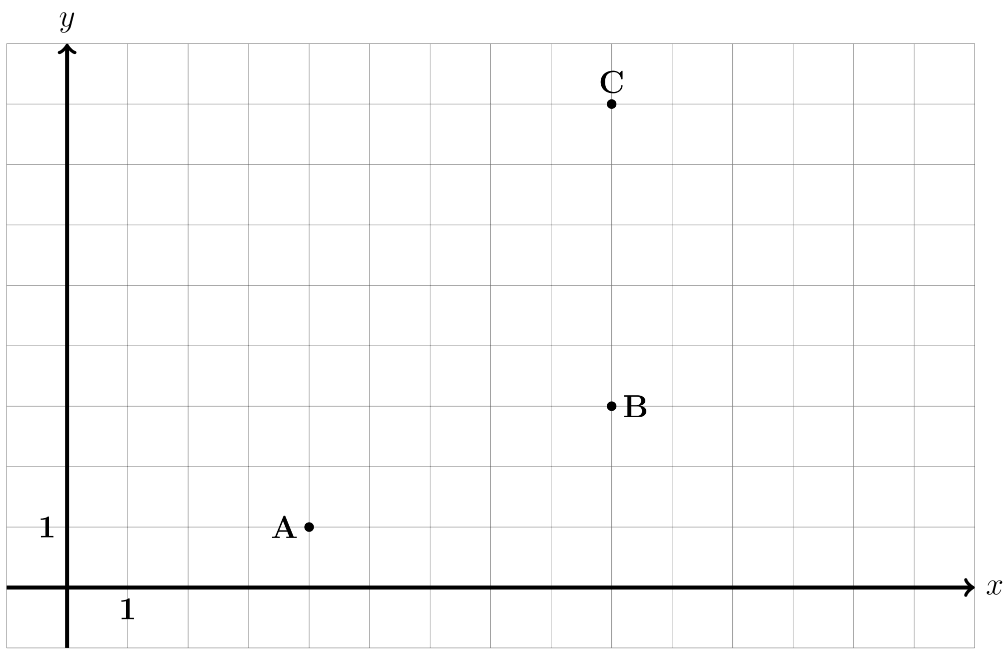
Problem 6 *Find the Euclidean distance $d_E(A, B)$ between the points A and B.*

$$d_E(A, B) =$$

Without doing any computations, put the correct sign, $>$, $<$, or $=$, between the distances below. Explain your choice.

$$d_E(A, B) \qquad\qquad d_{tc}(A, B)$$

Problem 7 *For the grid below, $a = s = 1$.*



Find the following taxicab distances.

$$d_{tc}(A, B) =$$

$$d_{tc}(A, C) =$$

$$d_{tc}(B, C) =$$

If we use the taxicab distance instead of the Euclidean one, would the triangle inequality hold for the triangle ABC?

For any two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in the coordinate plane, let us define the taxicab distance between them as follows.

$$d_{tc}(A, B) = |x_1 - x_2| + |y_1 - y_2| \tag{2}$$

Problem 8 *Find the taxicab distance between the points* $A = (-2, 7)$ *and* $B = (3, -5)$.

$$d_{tc}(A, B) =$$

Problem 9 *The taxicab distance between the points* A *and* B *is zero.*

$$d_{tc}(A, B) = 0$$

Can the points be different? Why or why not?

Note that the taxicab distance shares some basic properties with the Euclidean one. The distance from A to B equals the distance from B to A.

$$d_{tc}(A, B) = d_{tc}(B, A) \qquad\qquad d_E(A, B) = d_E(B, A)$$

In both cases, the distance between two points is zero if and only if the points coincide.

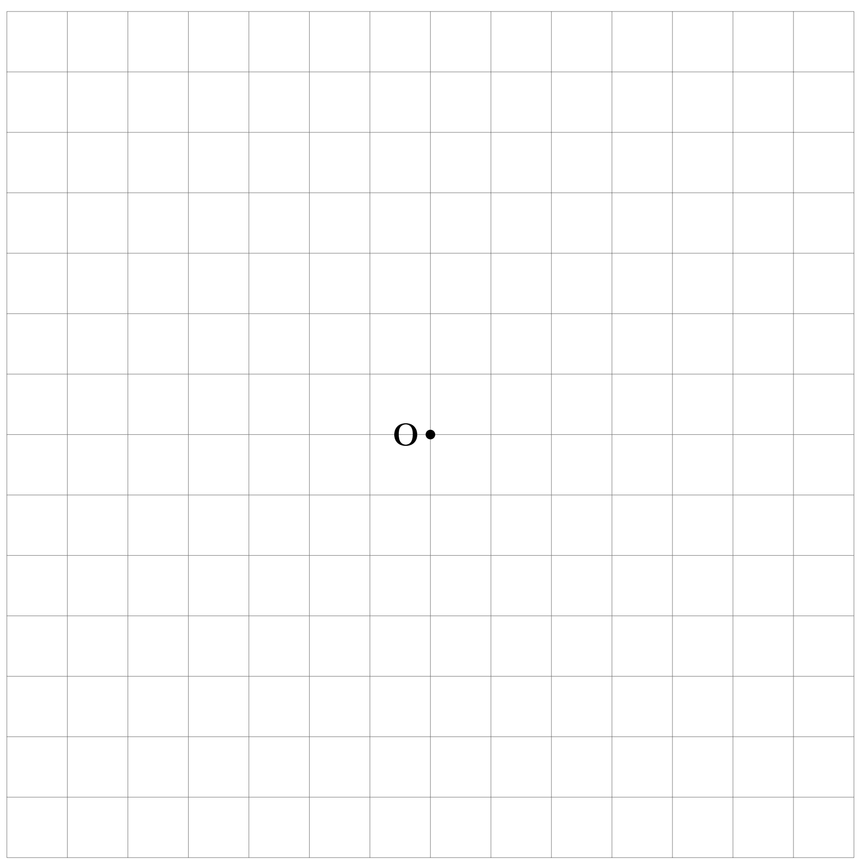
$$d_{tc}(A, B) = 0 \;\Leftrightarrow\; A = B \;\Leftrightarrow\; d_E(A, B) = 0$$

The distance between two different points is always positive.

$$A \neq B \;\Rightarrow\; d_{tc}(A,B) > 0 \;\text{ and }\; d_E(A,B) > 0$$

We have observed one difference between the distances in Problem 7. There, $d_{tc}(A,C) = d_{tc}(A,B) + d_{tc}(B,C)$. For the Euclidean distance, this means that the point B lies on the straight line AC between the points A and C, quite obviously not necessarily the case for the taxicab distance.

Problem 10 *On the grid below, mark all the points that have the taxicab distance 6 from the point O.*



Problem 11 *Give the definition of a circle of radius R centred at the point O in the space below.*

Was the figure constructed in Problem 10 a circle? Why or why not?

Problem 12 *Find the taxicab distance from the current position of the empty square to the lower-right corner of the 15 puzzle.*

1	**2**	**3**	**4**
5	**6**		**8**
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Finally, we have all the tools we need to prove that Sam Loyd's configuration is unsolvable.

Let $\mathcal{P}$ be a function that assigns each configuration $\mathcal{C}$ of the 15 puzzle one of the two values, either zero or one. Let us set $\mathcal{P}(\mathcal{C}) = 0$ if the the sum of the inversions of $\mathcal{C}$ plus the taxicab distance from its empty square position to the lower-right corner of the puzzle is an even number. Let us set $\mathcal{P}(\mathcal{C}) = 1$ otherwise. For example, let us take another look at the configuration $\mathcal{C}$ we have considered in Problems 2 and 12.

1	**2**	**3**	**4**
5	**6**		**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

The number of inversions of this configuration is 16. The taxicab distance from the empty square of the configuration to the lower-right corner is 3. The sum, $16 + 3 = 19$, is an odd number, so $\mathcal{P}(\mathcal{C}) = 1$.

Let us call a configuration $\mathcal{C}$ of the 15 puzzle *even*, if $\mathcal{P}(\mathcal{C}) = 0$ and let us call it *odd* otherwise. This way, all the configurations of the puzzle are split into two classes, even and odd.

Problem 13 *Is the following configuration*

1	**2**	**3**	**4**
5		**6**	**8**
9	**10**	**7**	**11**
13	**15**	**14**	**12**

even or odd? Try to answer this question without doing too many calculations. Hint: compare this configuration to that on page 10.

The following theorem answers the solvability question that has opened this mini-course.

Theorem 1 *Odd configurations of the 15 puzzle are not solvable.*

Indeed, Sam Loyd's configuration $\mathcal{C}$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
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has only one inversion, $(15, 14)$. The taxicab distance from the empty square position of the configuration to the lower-right corner is zero. Hence, $\mathcal{P}(\mathcal{C}) = 1 + 0 = 1$. The configuration is odd and thus, according to Theorem 1, has no solution.

Proof of Theorem 1 — Each move of the 15 puzzle is a transposition that swaps a square numbered 1 through 15 with the empty square. According to Theorem 1 from the previous handout[1], a transposition always changes the number of inversions of a permutation by an odd number.

Any move of the 15 puzzle changes the taxicab distance from the current empty square position to the lower-right corner by one. Hence, the sum of the changes of the number of the inversions and of the taxicab distance in consideration is always an even number.

[1] `http://www.math.ucla.edu/~radko/circles/lib/data/Handout-707-822.pdf`

The winning configuration of the 15 puzzle, the one corresponding to the trivial permutation of the sixteen element, is even. According to the above, it cannot be obtained from an odd configuration. □

Theorem 2 *Any even configuration of the 15 puzzle is solvable.*

Theorem 2 is not hard to prove using mathematical induction. We are not going to do it at the moment. The following theorem is much harder to prove.

Theorem 3 *Lengths of the optimal solutions of the 15 puzzle range from 0 to 80 single-tile moves.*