LAMC Beginners' Circle: Parity of a Permutation

Problems from Handout by Oleg Gleizer

Solutions by James Newton

1. *Take a two-digit number and write it down three times to form a six-digit number. For example, the two-digit number 26 gives rise to the six-digit number 262626. Prove that the resulting six-digit number is always divisible by 3, 7, 13, 37, 111 and 1443.*

   Solution. First we note that if we let A range from 1 to 9, and let B range from 0 to 9, then the decimal notation $ABABAB_{10}$ has the value $A \cdot 10^5 + B \cdot 10^4 + A \cdot 10^3 + B \cdot 10^2 + A \cdot 10 + B$.

   (To review decimal place-value notation, consult the handout from 29 September 2013, available online at ⟨ http://www.math.ucla.edu/∼radko/circles/lib/data/Handout-541-664.pdf ⟩.)

   Now we group together all the terms containing A and all the terms containing B. Then $ABABAB_{10}$ has the value $A(10^5 + 10^3 + 10) + B(10^4 + 10^2 + 1)$. Notice that the term in parentheses multiplied by A has a common factor of 10. Thus we can factor that out of the expression as well: $10A(10^4 + 10^2 + 1) + B(10^4 + 10^2 + 1)$. But wait! Now notice that in fact, we can factor out the term $10^4 + 10^2 + 1$ from the entire expression. So $ABABAB_{10}$ has the value $(10A + B)(10^4 + 10^2 + 1)$.

   If we switch back to decimal notation, we have $ABABAB_{10} = AB_{10} \cdot 10101_{10}$.

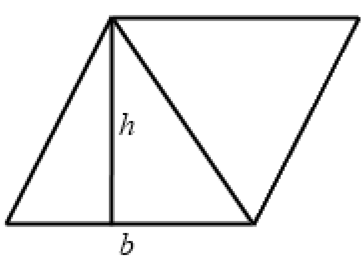
   Now suppose we have an integer n such that $ABABAB_{10}$ is divisible by n. If we wish to show that $ABABAB_{10}$ is *always* divisible by n (as indeed we do), then this property must be independent of our choice of A and B. Thus what we really need to show is that 10101_{10} is divisible by n.

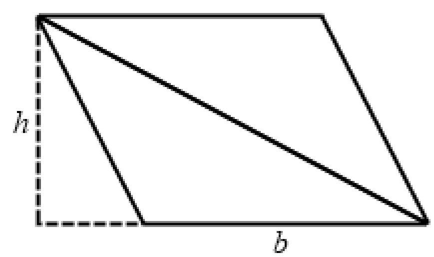
   If we factor 10101_{10} – that is, if we divide until we obtain only prime numbers – we see that $10101 = 3 \cdot 7 \cdot 13 \cdot 37$ (dropping the "$_{10}$" tag as it is unnecessary at this point). You can multiply this out yourself to verify the equality. Thus we know that 10101, and thus $ABABAB_{10}$, is always divisible by 3, 7, 13 and 37.

   Now notice that $111 = 3 \cdot 37$. Then because $ABABAB_{10}$ is always divisible by both 3 and 37, it is divisible by 111. Similarly, as $1443 = 3 \cdot 13 \cdot 37$, as $ABABAB_{10}$ is always divisible by 3, 13 and 37, it is also divisible by 1443. So any number of the form $ABABAB_{10}$ is always divisible by 3, 7, 13, 37, 111 and 1443. *Done.*

2. *Prove that in the Euclidean geometry, the area of a triangle is one half of the product of its base and height.*

   Solution. From any triangle, we can construct a parallelogram with the same values of base and height. To do this, we make a copy of the triangle and rotate it about an edge.





   The parallelogram has area bh, and is composed of two non-overlapping instances of our triangle. Thus the area of our triangle is half of this, or $\frac{1}{2}bh$. *Done.*

3. *Is it possible to have two triangles in the Euclidean plane such that every side of the first triangle is longer than every side of the second triangle, but the second triangle has a greater area? Why or why not?*

   Solution. The triangle with side lengths $100, \sqrt{2499}, \sqrt{2499}$ has area 50. A second triangle with shorter side lengths $12, \sqrt{61}, \sqrt{61}$ has area 60. This is because the first triangle, while having longer side lengths, has a very wide largest angle, and appears to be little more than a sliver. In general, one can construct a triangle with long sides and small area by taking a long side length and small height, then using the Pythagorean Theorem to find the other side lengths. *Done.*

4. *Find the order of the permutation $\sigma = (\ 5\ 1\ 4\ 3\ 2\ )$.*

   Solution. Compute successive powers of σ. Recall that we obtain σ^n by applying σ to the result of σ^{n-1}.

$$\begin{array}{rclclcl} \sigma^2 &=& \sigma\circ\sigma &=& (\ 5\ 1\ 4\ 3\ 2\ )\circ(\ 5\ 1\ 4\ 3\ 2\ ) &=& (\ 2\ 5\ 3\ 4\ 1\ ) \\ \sigma^3 &=& \sigma\circ\sigma^2 &=& (\ 5\ 1\ 4\ 3\ 2\ )\circ(\ 2\ 5\ 3\ 4\ 1\ ) &=& (\ 1\ 2\ 4\ 3\ 5\ ) \\ \sigma^4 &=& \sigma\circ\sigma^3 &=& (\ 5\ 1\ 4\ 3\ 2\ )\circ(\ 1\ 2\ 4\ 3\ 5\ ) &=& (\ 5\ 1\ 3\ 4\ 2\ ) \\ \sigma^5 &=& \sigma\circ\sigma^4 &=& (\ 5\ 1\ 4\ 3\ 2\ )\circ(\ 5\ 1\ 3\ 4\ 2\ ) &=& (\ 2\ 5\ 4\ 3\ 1\ ) \\ \sigma^6 &=& \sigma\circ\sigma^5 &=& (\ 5\ 1\ 4\ 3\ 2\ )\circ(\ 2\ 5\ 4\ 3\ 1\ ) &=& (\ 1\ 2\ 3\ 4\ 5\ ) = e \end{array}$$

   As $\sigma^6 = e$ and $\sigma^k \neq e$ for any $0 < k < 6$, by definition we have that the order of σ is $\texttt{ord}(\sigma) = 6$. *Done.*

5. *Without doing any more computations, find σ^{-1} and σ^{126} for the permutation $\sigma = (\ 5\ 1\ 4\ 3\ 2\ )$.*

   Solution. By the definition of the inverse of a permutation σ^{-1} is a permutation such that $\sigma\circ\sigma^{-1} = \sigma^{-1}\circ\sigma = e$. From our computations in Problem 4, we have already found that $\sigma\circ\sigma^5 = \sigma^6 = e$. Thus $\sigma^{-1} = \sigma^5 = (\ 2\ 5\ 4\ 3\ 1\ )$. In addition, $\sigma^{126} = (\sigma^6)^{21} = e^{21} = e = (\ 1\ 2\ 3\ 4\ 5\ )$. *Done.*

6. *Find $\sigma(2)$, $\sigma(3)$ and $\sigma(4)$ for $\sigma = (\ 5\ 1\ 4\ 3\ 2\ )$.*

   Solution. σ moves the element in the 2nd position to the 5th position, hence $\sigma(2) = 5$. Similarly, $\sigma(3) = 4$ and $\sigma(4) = 3$ because σ moves the element in the 3rd position to the 4th position, and vice versa. *Done.*

NOTE 1 ON INVERSIONS. As discussed in the handout, inverses and inversions are very different. The **inverse** of a permutation σ is another permutation, one which undoes the effect of σ. For a permutation σ, there is only *one* inverse σ^{-1}. (Can you prove why?) However, there may be (and often are) *multiple* inversions. An **inversion** of a permutation is a pair of positions (i, j) such that before σ is applied, i comes after j, but after σ is applied, i comes before j. As we usually denote the positions with integers, this can be restated: an inversion of σ is a pair (i, j) such that i comes before j in the expression of σ, but $i > j$.

NOTE 2 ON INVERSIONS. Note that an inversion is expressed as a *pair* of values; the comma is important! (i, j) denotes an inversion of the ith and jth positions. $(\ i\ j\ )$, on the other hand, is a transposition – a *permutation* which switches the place of the ith and jth positions.

NOTE 3 ON INVERSIONS. There is a way to parse a permutation to write down all inversions. Let $\sigma = (\ a_1\ a_2\ \cdots\ a_n\ )$ be a permutation on n elements. (So the numbers a_i range from 1 to n, with no duplicates; $a_i \neq a_j$ for $i \neq j$.) We begin at a_1, and consider all a_i with $i > 1$. If $a_j < a_i$, then (a_i, a_j) is an inversion of σ. Once we have written all inversions involving a_1, we move on to a_2 and repeat until we reach the end.

EXAMPLE. Find all inversions of the permutation $\sigma = (\ 5\ 1\ 4\ 3\ 2\ )$.

Solution. We begin at the first position: 5. Looking to the right, we see that $1 < 5$, so $(5, 1)$ is an inversion. Continuing, we have $4 < 5$, so $(5, 4)$ is an inversion. Continuing, we have $3 < 5$, so $(5, 3)$ is an inversion. Finally, we reach the end and have $2 < 5$, so $(5, 2)$ is an inversion. These are all of the inversions involving 5. So we move to the next position: 1. Well, 1 is the lowest possible value for a position, so nothing to the right of 1 will be less. So we can continue to the next position: 4. Looking to the right, we have both $3 < 4$ and $2 < 4$. Hence $(4, 3)$ and $(4, 2)$ are inversions. There are no more inversions involving 4. Move to the next position: 3. Looking to the right, $2 < 3$ and $(3, 2)$ is an inversion. Finally we have reached the last position and are done.

Thus the inversions of σ are $(5, 1)$, $(5, 4)$, $(5, 3)$, $(5, 2)$, $(4, 3)$, $(4, 2)$ and $(3, 2)$.

7. *Write down all other inversions of the permutation $\sigma = (\ 5\ 1\ 4\ 3\ 2\ )$.*

   Solution. See the Notes and Example above. *Done.*

8. *What is the sign of the trivial permutation?*

   Solution. The trivial permutation e, by definition, does not change the order of any elements. Hence there are no inversions. Thus $\texttt{sgn}(e) = (-1)^0 = 1$. *Done.*

9. *Compute the following signs of 4-element permutations.*
   - $\sigma = (\ 3\ 1\ 4\ 2\ )$
     Solution. The inversions of σ are $(3,1)$, $(3,2)$ and $(4,2)$. Then $\mathtt{sgn}(\sigma) = (-1)^3 = -1$,
   - $\sigma = (\ 3\ 2\ 4\ 1\ )$
     Solution. The inversions of σ are $(3,2)$, $(3,1)$, $(2,1)$ and $(4,1)$. Then $\mathtt{sgn}(\sigma) = (-1)^4 = 1$. *Done.*

10. *What is the sign of the permutation corresponding to the following configuration of the 15-puzzle? (Recall the empty square is labelled as position 16.)*

    Solution. Let $\sigma = (\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 16\ 11\ 13\ 15\ 14\ 12\ )$, corresponding to the above configuration. The inversions of σ are $(16,11)$, $(16,13)$, $(16,15)$, $(16,14)$, $(16,12)$, $(13,12)$, $(15,14)$, $(15,12)$ and $(14,12)$. There are nine inversions, thus $\mathtt{sgn}(\sigma) = (-1)^9 = -1$. *Done.*

11. *What is the sign of the following transpositions?*
    - $(\ 5\ 2\ )$ *on a set of five elements*
      Solution. On a set of five elements, $(\ 5\ 2\ )$ expands to $(\ 1\ 5\ 3\ 4\ 2\ )$ (recall that if a position does not appear in our notation of a permutation, it is unaffected by the permutation). The inversions of $(\ 1\ 5\ 3\ 4\ 2\ )$ are $(5,3)$, $(5,4)$, $(5,2)$, $(3,2)$ and $(4,2)$; hence $\mathtt{sgn}(\ 5\ 2\ ) = \mathtt{sgn}(\ 1\ 5\ 3\ 4\ 2\ ) = (-1)^5 = -1$.
    - $(\ 5\ 2\ )$ *on a set of six elements*
      Solution. On a set of six elements, $(\ 5\ 2\ )$ expands to $(\ 1\ 5\ 3\ 4\ 2\ 6\ )$. The inversions of $(\ 1\ 5\ 3\ 4\ 2\ 6\ )$ are $(5,3)$, $(5,4)$, $(5,2)$, $(3,2)$ and $(4,2)$; hence $\mathtt{sgn}(\ 5\ 2\ ) = \mathtt{sgn}(\ 1\ 5\ 3\ 4\ 2\ 6\ ) = (-1)^5 = -1$.
      Notice that the inclusion of a sixth element in the set has not affected the sign of the transposition, or the inversions. Can you see why?
    - $(\ 6\ 3\ )$ *on a set of seven elements*
      Solution. On a set of seven elements, $(\ 6\ 3\ )$ expands to $(\ 1\ 2\ 6\ 4\ 5\ 3\ 7\ )$. The inversions of $(\ 1\ 2\ 6\ 4\ 5\ 3\ 7\ )$ are $(6,4)$, $(6,5)$, $(6,3)$, $(4,3)$ and $(5,3)$; hence $\mathtt{sgn}(\ 6\ 3\ ) = \mathtt{sgn}(\ 1\ 2\ 6\ 4\ 5\ 3\ 7\ ) = (-1)^5 = -1$. *Done.*

12. *Represent the transposition* $(\ 6\ 3\ )$ *as a product of adjacent transpositions.*

    Solution. On a set of at least six elements, this transposition swaps the third and sixth elements. We will achieve this by "pushing" (through the use of repeated adjacent transpositions) the third element to the sixth position, then "pushing" the sixth element back to the third position.

$$\begin{array}{rcl} & & (\ 1\ 2\ 3\ 4\ 5\ 6\ \cdots\ ) = e \\ & & (\ 1\ 2\ 4\ 3\ 5\ 6\ \cdots\ ) = (\ 4\ 3\ ) \\ & & (\ 1\ 2\ 4\ 5\ 3\ 6\ \cdots\ ) = (\ 5\ 4\ )\circ(\ 4\ 3\ ) \\ & & (\ 1\ 2\ 4\ 5\ 6\ 3\ \cdots\ ) = (\ 6\ 5\ )\circ(\ 5\ 4\ )\circ(\ 4\ 3\ ) \\ & & (\ 1\ 2\ 4\ 6\ 5\ 3\ \cdots\ ) = (\ 5\ 4\ )\circ(\ 6\ 5\ )\circ(\ 5\ 4\ )\circ(\ 4\ 3\ ) \\ (\ 6\ 3\ ) & = & (\ 1\ 2\ 6\ 4\ 5\ 3\ \cdots\ ) = (\ 4\ 3\ )\circ(\ 5\ 4\ )\circ(\ 6\ 5\ )\circ(\ 5\ 4\ )\circ(\ 4\ 3\ ) \end{array}$$

    The number of adjacent transpositions is 5, which is odd. Note that we knew ahead of time that the number of transpositions would be odd, thanks to the Lemma in the handout. *Done.*

13. - *Find the sign of the permutation μ acting on a set of five elements.*
      Solution. On a set of five elements, μ expands to $(\ 3\ 2\ 4\ 1\ 5\ )$. The inversions of $(\ 3\ 2\ 4\ 1\ 5\ )$ are $(3,2)$, $(3,1)$, $(2,1)$ and $(4,1)$; hence $\mathtt{sgn}(\ 3\ 4\ 1\ ) = \mathtt{sgn}(\ 3\ 2\ 4\ 1\ 5\ ) = (-1)^4 = 1$. *Done.*
    - *Find the product* $(\ 5\ 1\ )\circ\mu$. On a set of five elements, μ swaps the first and last elements. Then $(\ 5\ 1\ )\circ(\ 3\ 2\ 4\ 1\ 5\ ) = (\ 5\ 2\ 4\ 1\ 3\ ) = (\ 5\ 4\ 1\ 3\ )$ in condensed notation. *Done.*
    - *Find the sign of the permutation* $(\ 5\ 1\ )\circ\mu$. The inversions of $(\ 3\ 2\ 4\ 1\ 5\ )$ are $(3,2)$, $(3,1)$ and $(4,1)$. Then $\mathtt{sgn}((\ 5\ 1\ )\circ\mu) = (-1)^3 = -1$. *Done.*

14. *Is it possible to cut some circles out of a square with side length one so that the sum of the circles' diameters is more than 2014? Why or why not?*

Solution. YES. Consider a square with side length s, and a circle *inscribed* in the square; that is, draw a circle inside the square such that it is tangent to all four sides. Then the diameter of the circle is equal to the square's side length s. Now take the square with side length 1, and divide each side into N equal parts, with $N > 2014$. By making a grid from these divisions, we have N^2 smaller squares, each with side length $1/N$. Then if we inscribe circles in each of these squares, the diameter of each circle is $1/N$. But we have N^2 such circles, so the sum of their diameters is $N^2 \cdot (1/N) = N^2/N = N > 2014$. *Done.*

15. *Alice and Bob take turns putting same-sized coins on a rectangular table. The first person unable to place a coin on the table without making it overlap with other coins loses. Find the winning strategy for the game.*

Solution. Move first, and place a coin in the center of the table. Whenever your opponent places a coin on the table, on your next move place a coin at its reflection through the center coin. So for example, if my opponent placed a coin at the left end of the table, I would place a coin at the right end of the table. If my opponent placed a coin at the top-right corner of the table, I would place a coin at the bottom-left corner of the table. And so on.

Why is this a winning strategy? A winning strategy for this game is an algorithm to place a coin given our opponent's last move, so that we wind up winning the game. So we need to be guaranteed a place to put a coin, based on our opponent's last move. We exploit the symmetry of the table. If we follow the above strategy, and we only place a coin at the opposite location of our opponent's coin, then if our opponent has room to place a coin, we are guaranteed that same room at the corresponding location, reflected through the center. The only location where this does not hold is the center itself (its reflection through the center is itself), but we have taken care of this problem by choosing the center first. *Done.*

This winning strategy only works if one moves first; however, this does not mean that whoever moves first is guaranteed a win, nor does it mean that whoever goes second will necessarily lose. Consider if the second player follows the same principle of "mirroring" the first player's moves. As long as the first player does not choose the center, then the second player is similarly always guaranteed space to place a coin. However, as this is contingent upon the first player not choosing the center, it does not count as a winning strategy.

16. *Every tenth mathematician is a philosopher. Every hundredth philosopher is a mathematician. Are there more philosophers or mathematicians? How much more?*

Solution. Let M denote the number of mathematicians, P denote the number of philosophers, and B denote the number of people who are both mathematicians and philosophers. The problem says that every tenth mathematician is (also) a philosopher, thus $B = M/10$ and $M = 10B$. Similarly, $P = 100B$. Then dividing, we have $P/M = 10$. So there are more philosophers than mathematicians, and there are in fact ten times as many philosophers as mathematicians! *Done.*

17. *Alice counted all the natural numbers from* 1 *to* 2014 *that are multiples of* 8*, but not multiples of* 9*. Bob counted all the natural numbers from* 1 *to* 2014 *that are multiples of* 9*, but not multiples of* 8*. Who has the greater number, Alice or Bob?*

Solution. Recall the **floor function** $\lfloor x \rfloor$, which returns the greatest integer less than or equal to x. So for example, $\lfloor 8.5 \rfloor = 8$, $\lfloor \pi \rfloor = 3$, $\lfloor 4 \rfloor = 4$, and $\lfloor -1.3 \rfloor = -2$. Also recall that the count of natural numbers from 1 to n that are multiples of m is given by $\lfloor n/m \rfloor$. This can be computed by dividing n by m in the usual way, and disregarding the remainder.

Then Alice's count will be $\lfloor 2014/8 \rfloor - \lfloor 2014/72 \rfloor$ (Can you see why?), and Bob's count will be $\lfloor 2014/9 \rfloor - \lfloor 2014/72 \rfloor$. We can compute these directly, and find that Alice's count is 224, which is greater than Bob's count 196. In fact, we can make our work easier by noticing that each count involves subtracting the same amount $\lfloor 2014/72 \rfloor$, and so any inequality between the two counts is preserved if we ignore this term. Then $\lfloor 2014/8 \rfloor = 251 > 223 = \lfloor 2014/9 \rfloor$, and Alice's count is greater. *Done.*

We might think that we can make our life even more simple by not having to compute either of these floored values. After all, with Alice's count, 2014 is being divided by 8, and in Bob's count the same number 2014 is being divided by a larger number, 9. So clearly Alice's count is greater, right? Not necessarily. The floor function makes this invalid. While 2014/8 is certainly greater than 2014/9, it may not be greater by a whole integer part. For example, consider $31/8 > 31/9$, but $\lfloor 31/8 \rfloor = \lfloor 31/9 \rfloor = 3$. So we do need to compute these floored values to be certain.

18. • *In the decimal place-value system, count the six-digit numbers that have at least one even digit.*

  Solution 1. (Inclusion-Exclusion Principle) First we do a naive count of numbers with at least one even digit. We fix some digit to be even, hence there are five choices $(0,2,4,6,8)$. We then let the other five digits be any of the decimal digits, hence 10^5 choices. But we had six positions which feasibly could have been fixed. So the count we have so far is $6 \cdot 5 \cdot 10^5$. But if there are multiple even digits, then we have counted them multiple times.

  Suppose we have at least r even digits. Each of the r fixed digits has five choices, being even. Then let the other $6-r$ digits be any of the decimal digits, hence 10^{6-r} choices. Recall there are $\binom{6}{r} = \frac{6!}{r!(6-r)!}$ ways to choose r positions to fix. So the naive count we get for at least r even digits is $\binom{6}{r} 5^r 10^{6-r}$.

  Applying the Inclusion-Exclusion Principle to the original problem, our count is $6 \cdot 5 \cdot 10^5 - 15 \cdot 5^2 \cdot 10^4 + 20 \cdot 5^3 \cdot 10^3 - 15 \cdot 5^4 \cdot 10^2 + 6 \cdot 5^5 \cdot 10 - 5^6 = 984375$. *Done.*

  Solution 2. (Modification of Solution 1) What if, instead of doing naive counts and applying the Inclusion-Exclusion Principle, we modified our counting slightly? In particular, when we fix r digits to be even, instead of letting the other $6-r$ digits be any decimal digit, we made sure they were odd? Then there are $\binom{6}{r} 5^r 5^{6-r} = \binom{6}{r} 5^6$ six-digit numbers with *exactly* r even digits. We can now add these values for $1 \leq r \leq 6$, since there are no longer any overlaps. So our final count is $6 \cdot 5^6 + 15 \cdot 5^6 + 20 \cdot 5^6 + 15 \cdot 5^6 + 6 \cdot 5^6 + 5^6 = (6+15+20+15+6+1)5^6 = 63 \cdot 5^6 = 984375$. *Done.*

  Solution 3. (Efficient Solution) Frankly, this is a lot of addition and coming up with multiple terms. Let's try a different approach. Notice that the opposite of having *at least one* even digit is having *no* even digits. So to count the six-digit numbers with at least one even digits, we can find the total amount of six-digit numbers and subtract the count of six-digit numbers with all odd digits. Well, there are 10^6 six-digit numbers. If a six-digit number has all odd digits, then for each of 6 digits there are 5 possibilities, hence 5^6 six-digit numbers with all odd digits. So our count is $10^6 - 5^6 = 2^6 5^6 - 5^6 = (2^6 - 1)5^6 = 63 \cdot 5^6 = 984375$. *Done.*

  REMARK. Solution 3 is the preferred solution, as it involves the least amount of side computation. Solutions 1 and 2 were included to review the Inclusion-Exclusion Principle.

  • *Solve the same problem for hexadecimals.*

  Solution. We use the method of Solution 3 above. There are 16 hexadecimal digits, and 8 even digits (hence 8 odd digits). Then our count is $16^6 - 8^6 = 2^6 8^6 - 8^6 = (2^6 - 1)8^6 = 63 \cdot 8^6 = 16515072$. *Done.*

19. *The father of a 5-year-old boy is 32. When would the man be ten times older than his son?*

Let x denote years from the present. Then we seek x such that $32 + x = 10(5 + x) = 50 + 10x$. Then $-9x = 18$ and $x = -2$. Oh, dear. A negative number? Well, recall that we defined x to be years from the present, looking to the future. A negative value suggests that we look in the opposite direction, to the past. So what our solution means is that two years ago, the man was ten times older than his son. Indeed, two years ago, the boy was 3 years old and his father was 30 years old. *Done.*

20. *Simplify the following algebraic expressions.*

$$\frac{(a+1)(2a+1)-a}{a(2a+1)+a+1} = \frac{2a^2+a+2a+1-a}{2a^2+a+a+1} = \frac{2a^2+2a+1}{2a^2+2a+1} = 1$$

$$\frac{a(a+b)-b}{a+(a+b)(a-1)} = \frac{a^2+ab-b}{a+a^2-a+ab-b} = \frac{a^2+ab-b}{a^2+ab-b} = 1$$

$$\frac{a(2a-1)-(a-1)}{(a-1)(2a-1)+a} = \frac{2a^2-a-a+1}{2a^2-a-2a+1+a} = \frac{2a^2-2a+1}{2a^2-2a+1} = 1$$

21. *Two pirates have to share a treasure. The treasure is made of objects very hard to compare, gemstones, pearls, gold and silver coins of various denomination and value, jewellery, silks, and so forth. The pirates are very violent. If one suspects the other of trying to take more than his fair share, a fight to the death will ensue. The pirates' tradition does not allow to break, cut, melt, or otherwise split a piece of booty into parts. (It is considered a bad omen.) How can the pirates divide the treasure in such a way that will keep both of them happy for sure and prevent bloodshed?*

Solution. Have one pirate divide the treasure into two piles that he deems to be of equal worth. Then have the second pirate choose the pile he deems to be of greater worth. Then the second pirate is happy, since he gets a pile of greater or equal value to him than the value of the one going to the other pirate. The first pirate is also happy, since he is fine with either pile: they are worth the same to him. *Done.*

Now the tough one: Solve the same problem for three pirates.

Solution. (Selfridge-Conway Discrete Procedure) Let's label the three pirates $P1, P2, P3$. We begin again with $P1$ dividing the booty into three piles that he deems to be of equal worth. Let's call these piles A, B and C. Now consider $P2$. If $P2$ agrees with $P1$ in that the three piles are of equal worth, then $P3$ chooses first. $P3$ is happy because his pile is of greater or equal value than that of the other piles. $P2$ chooses next and is happy because all the piles are worth the same to him, as is $P1$.

Now suppose $P2$ decides that A and B are of equal value, but C is of lesser value. Then again, let $P3$ choose first, so $P3$ is happy. If $P3$ chooses C, then all pirates are happy as in the previous case. If $P3$ does not choose C, then $P2$ chooses second and also does not choose C, hence is happy because to him, he has the piece of greater value. $P1$ is left with C, and is happy because all piles were worth the same to him.

Now suppose $P2$ decides that A is worth the most, and B is of greater or equal value than C. Then $P2$ divides A into $A1$ and $A2$ such that $A1$ is worth the same as B, to him. Now $P3$ is allowed to choose among $A1$, B and C. $P2$ then chooses from the remainder, with the restriction that if $P3$ did not choose $A1$, then $P2$ must take it. $P1$ receives what is left over. $P3$ is happy as he chose first, $P2$ is happy because $A1$ has the maximum value of his options, and $P1$ is happy because he receives either B or C, which are of equal value to him and worth more than $A1$. So far, so good.

Now all that remains is to divide $A2$ so everyone is happy. One of $P2$ and $P3$, we know, has chosen the piece $A1$. We relabel so that this pirate is PA and the other is PB. ($P1$ is still $P1$.) Now PB divides $A2$ into three piles of equal value to him. PA chooses a pile first, then $P1$ and finally PB. $P1$ is happy with this because anything in addition to his previous allotment is more than one-third of the total value in his estimate. PA is happy because he is the first to choose from this selection, and PB is happy because all subdivisions of $A2$ are of equal value to him. *Done.*