LAMC Beginners' Circle **April 27, 2014**

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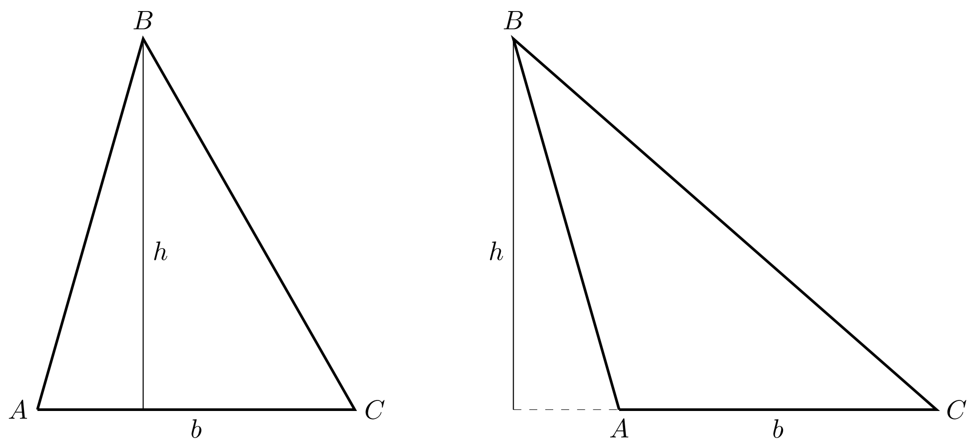
Warm-up

Problem 1 *Take a two-digit number and write it down three times to form a six-digit number. For example, the two-digit number 26 gives rise to the six-digit number 262626. Prove that the resulting six-digit number is always divisible by 3, 7, 13, 37, 111, and 1,443.*

Problem 2 *Prove that in the Euclidean geometry, the area of a triangle is one half of the product of its base and height. Consider both cases depicted below.*



Problem 3 *Is it possible to have two triangles in the Euclidean plane such that every side of the first triangle is longer than every side of the second triangle, but the second triangle has a greater area? Why or why not?*

Back to the 15 puzzle

Problem 4 *Find the order of the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$.

Problem 5 *Without doing any more computations, find the following for the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ *from Problem 4.*

$$\sigma^{-1} =$$

$$\sigma^{126} =$$

Parity of a permutation

If a permutation σ moves the element in the position i to the position k, we write $\sigma(i) = k$. Let us consider the permutation σ from Problems 4 and 5 one more time. It moves the fifth element to the first position, so $\sigma(5) = 1$. It moves the first element to the second position, so $\sigma(1) = 2$.

Problem 6 *For the permutation* σ *from Problems 4 and 5, find the following.*

$$\sigma(2) =$$

$$\sigma(3) =$$

$$\sigma(4) =$$

If $i < j$, but $\sigma(i) > \sigma(j)$, then the pair (i, j) is called an *inversion* of the permutation σ. In other words, a inversion of a permutation is a smaller number moved to the right of a larger number (or a larger number moved to the left of a smaller number). For example, the permutation $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ from Problems 4, 5, and 6 moves 5 to the first position, so $(5, 1)$, $(5, 4)$, $(5, 3)$, and $(5, 2)$ are all inversions of σ.

Note 1 *Although the words "inverse" and "inversion" are very similar, the notions of an inverse of a permutation and an inversion of a permutation are very different! An inverse of a permutation σ is the permutation σ^{-1} that undoes what the original permutation σ does. The inversion of a permutation σ is a disorder the permutation σ creates.*

Problem 7 *Write down all other inversions of the permutation* $\sigma = \begin{pmatrix} 5 & 1 & 4 & 3 & 2 \end{pmatrix}$.

The *sign* of a permutation is defined according to the following formula.

$$\operatorname{sgn}(\sigma) = (-1)^{N(\sigma)} \tag{1}$$

where $N(\sigma)$ is the number of inversions of the permutation σ. For example, the total number of inversions of the permutation σ from Problems 4, 5, 6, and 7 is seven (check it!), so $\operatorname{sgn}(\sigma) = (-1)^7 = -1$.

Problem 8 *What is the sign of the trivial permutation?*

$$sgn(e) =$$

Problem 9 *Find the signs of the following permutations.*

$$sgn\left(3\ 1\ 4\ 2\right) =$$

$$sgn\left(3\ 2\ 4\ 1\right) =$$

Problem 10 *What is the sign of the permutation corresponding to the following configuration of the 15 puzzle? (Remember, the empty square is considered as the 16th tile.)*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**		**11**
13	**15**	**14**	**12**

Recall that a transposition (ji) is a permutation that changes the positions of only two elements, i-th and j-th.

Theorem 1 *The sign of any transposition is* -1.

Before giving Theorem 1 a formal proof, let us check a few cases.

Problem 11 *What is the sign of the transposition* $\sigma = (52)$ *acting on a set of five elements?*

$$sgn(\sigma) =$$

What is the sign of the transposition $\sigma = (52)$ *acting on a set of six elements?*

$$sgn(\sigma) =$$

What is the sign of the transposition $\sigma = (63)$ *acting on a set of seven elements?*

$$sgn(\sigma) =$$

To prove Theorem 1, let us first observe that a transposition of two neighbouring elements, called an *adjacent transposition*, always changes the number of inversions by one. Let us consider the transposition $\delta = (i+1, i)$. All the elements except for the $i+1$-st that formed inversions with the i-th element still form inversions with it when it moves to the $i+1$-st position. All the elements except fore the i-th that formed inversions with the $i+1$-st one keep doing so when the latter moves one position to the left. If the pair $(i, i+1)$ formed an inversion, δ removes it. If the pair formed no inversion, δ creates one.

The following Lemma finishes the proof of Theorem 1.

Lemma 1 *Any transposition can be realized as a product of an odd number of adjacent transpositions.*

Proof — Consider the transposition (ji) where $j > i+1$. The following product of $j-i-1$ adjacent transpositions

$$(j-1, j-2) \circ \ldots \circ (i+2, i+1) \circ (i+1, i)$$

moves the i-th element to the $j-1$-st position one step at a time. The adjacent transposition

$$(j, j-1)$$

swaps it with the j-th element. Finally, the following product of $j-i-1$ adjacent transpositions

$$(i+1, i) \circ (i+2, i+1) \circ \ldots \circ (j-1, j-2)$$

moves the element that was originally in the j-th position to the i-th. This way, any transposition (ji) where $j > i+1$ can be represented as a product of $2(j-i-1)+1$ adjacent transpositions. □

Example 1

$$(52) = (32) \circ (43) \circ (54) \circ (43) \circ (32)$$

Problem 12 *Represent the transposition* (63) *as a product of adjacent transpositions.*

$$(63) =$$

Is the number of the adjacent transpositions odd or even?

The permutations that have the sign 1 are called *even*. The permutations that have the sign -1 are called *odd*. This way, all permutations are split into two classes. A class of a permutation is called its *parity*. Theorem 1 proves that transpositions are odd permutations and that multiplying a permutation by a transposition changes the parity of the former.

Problem 13

- *Find the sign of the permutation* $\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$ *acting on a set of five elements.*

$$sgn(\mu) =$$

- *Find the product* $(51) \circ \mu$.

$$(51) \circ \mu =$$

- *Find the sign of the permutation* $(51) \circ \mu$.

$$sgn\left((51) \circ \mu\right) =$$

Note that Theorem 1 gives a different way to compute the sign of a permutation. Instead of counting inversions, let us decompose the permutation into a product of transpositions. Then the sign of the transposition is

$$(-1)^{\text{the number of transpositions in the product}}. \tag{2}$$

Various representations of a permutation as a product of transpositions can have different length, but they always have the same parity.

Every move of the 15 puzzle is a transposition of a special type. You swap a square numbered one through fifteen with the empty square (originally in the 16th position). This observation alone is not enough to prove that the 15 puzzle configuration suggested by Sam Loyd has no solution.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

We need one more tool, called the taxicab geometry. We will study it next time. This time, if you are finished doing all the above ...

... it's time for more problems!

Problem 14 *Is it possible to cut some circles out of a square with the side length one so that the sum of the circles' diameters is more than 2014? Why or why not?*

Problem 15 *Alice and Bob take turns putting coins of the same size on a rectangular table. The first person unable to place a coin on the table without making it overlap with other coins loses. Find the winning strategy for the game.*

Problem 16 *Every tenth mathematician is a philosopher. Every hundredth philosopher is a mathematician. Are there more philosophers or mathematicians? How much more?*

Problem 17 *Alice counted all the natural numbers from 1 to 2014 that are multiples of 8, but not multiples of 9. Bob counted all the natural numbers from 1 to 2014 that are multiples of 9, but not multiples of 8. Who got a greater number, Alice or Bob?*

Problem 18 *In the decimal place-value system, count the number of the six-digit numbers that have at least one even digit.*

Solve the same problem for hexadecimals.

Problem 19 *The father of a 5-year-old boy is 32. When would the man be ten times older than his son?*

Problem 20 *Simplify the following algebraic expressions.*

a. $$\frac{(a+1)(2a+1)-a}{a(2a+1)+a+1} =$$

b. $$\frac{a(a+b)-b}{a+(a+b)(a-1)} =$$

c. $$\frac{a(2a-1)-(a-1)}{(a-1)(2a-1)+a} =$$

Problem 21 *Two pirates have to share a treasure. The treasure is made of objects very hard to compare, gemstones, pearls, gold and silver coins of various denomination and value, jewellery, silks, and so forth. The pirates are very violent. If one suspects the other of trying to take more than his fair share, a fight to the death will ensue. The pirates' tradition does not allow to break, cut, melt, or otherwise split a piece of booty into parts. (It is considered a bad omen.) How can the pirates divide the treasure in such a way that will keep both of them happy for sure and prevent bloodshed?*

Solve the same problem for three pirates.