# Factorizations

LA Math Circle | Advanced Group

April 20, 2014

DIRECTIONS. For your solutions to these problems, you may only rely on basic algebra (and facts like $x^2 \geq 0$ for all $x$) or on problems that you have already solved. You may not, for example, rely on a theorem you found in a book, or something like the quadratic formula (unless you prove it in the course of your solution)

1. **(Warm-Up)** Verify the following four formulas related to factorization:

   (a) $a^2 - b^2 = (a-b)(a+b)$

   (b) $a^2 + b^2 = (a+b)^2 - 2ab$

   (c) $a^3 - b^3 = (a-b)(a^2 + ab + b^{2)}$

   (d) $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

2. If $\frac{1}{a+c} = \frac{1}{a} + \frac{1}{c}$, find $(\frac{a}{c})^3$.

3. Given that $123456 = 15241383936$, find $123457^2$. (Remember, no calculators allowed!)

4. What is the sum of the prime factors of $2^{16} - 1$?
(Please do this without grinding out $2^{16}$!)

5. **Without** solving the following equation, find the sum of the squares of its roots.

$$2x^2 - 3x - 4 = 0$$

(Hint: Use a theorem you proved in a recent handout...)

6. Find four nontrivial (not 1) factors in terms of $x$ and $y$ whose product is $8^{2x} - 27^{2y}$.

7. If $r$ and $s$ are the roots of $x^2 + px + q = 0$,then find each of the following in terms of $p$ and $q$.

   (a) $r^2 + s^2$

(b) $r - s$
(Do you see any significance to your answer?)

(c) $r^2s + rs^2$

(d) $r^4 + s^4$

8. Find two four-digit numbers whose product is $4^8 + 6^8 + 9^8$.
(Hint: You'll need to factor the expression first.)

9. Prove that all numbers that can be represented by $n^3 - n$, where $n$ is an integer, are divisible by 6. Can you also prove the converse?