

LEVERS AND BARYCENTRIC COORDINATES (PART 2)

LAMC INTERMEDIATE - 4/20/14

This handout (Part 2) continues the work we started last week when we discussed levers. This week we will focus on Barycentric coordinates and proving that the three medians of a triangle intersect at a single point. Many problems are taken from previous math circle handouts.

WARM-UP

(1) Write down a bijection $f : \mathbb{Z} \rightarrow \mathbb{N}$ from the integers to the natural numbers, and explicitly define its inverse.

(2) Suppose we have a set A and a function $f : A \rightarrow A$ which is one-to-one but not onto. What can we say about A ?

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- (3) Suppose we have infinitely many pigeons and seven pigeonholes, prove that at least one pigeonhole must contain infinitely many pigeons.

- (4) Prove that for a given decimal number $x = .a_1a_2a_3\cdots$ there is a sequence of rational numbers converging to x (write down this sequence explicitly!). What are the first few terms of this sequence if $x = \pi - 3$?

BARYCENTRIC COORDINATES

Suppose that x is between 0 and 1,

$$0 < x < 1.$$

Let $[0, 1]$ be our lever (so it is a lever of total length 1) and our pivot be at the point x . Weights w_0 and w_1 , placed at 0 and 1 respectively, which make the lever balanced, are called the **Barycentric Coordinates** of x , denoted $(w_0 : w_1)$.

(1) Suppose $x = \frac{1}{4}$.

(a) Find w_1 if $w_0 = 3$.

(b) Find w_1 if $w_0 = 6$.

(c) Find w_1 if $w_0 = 60$.

(d) Find w_1 if $w_0 = 120$.

(e) Find w_1 if $w_0 = 5$.

(2) In this problem we show that Barycentric Coordinates are invariant under scaling.

(a) Calculate the ratio $\frac{w_1}{w_0}$ for (a) - (e) in the previous problem.

(b) Prove that if $(w_0 : w_1)$ are barycentric coordinates for a point x then so are $(cw_0 : cw_1)$ for any non-zero number c .

(c) Prove also that if $(w_0 : w_1)$ and $(m_0 : m_1)$ have the same ratio, then they are coordinates for the same point x .

(d) Note that we can make the coordinates of a point unique by requiring $w_0 + w_1 = 1$, in which case the coordinates are no longer projective. In this case, what are the Barycentric Coordinates of the point $x = \frac{1}{4}$ from the previous problem?

(3) Assuming $w_0 + w_1 = 1$, find the Barycentric Coordinates of the following points.

(a) $\frac{1}{4}$

(b) $\frac{3}{4}$

(c) $\frac{2}{5}$

(d) $\frac{1}{7}$

(e) .99

(f) .8

(g) 1

(h) 0

(4) So far we just have Barycentric Coordinates on the interval $[0, 1]$, but we want to extend them to the entire number line!

(a) How can you balance the lever if you must put two weights on the same side of the pivot? (Hint: use a concept from the handout last week)

(b) Using your method from (a), find the Barycentric Coordinates of the following points.

(i) 3

(ii) -3

(iii) 4

(iv) 100

(v) -17

(vi) 3.25

(vii) $\frac{3}{5}$

(5) Notice that the points 0 and 1 really aren't all that important. In fact, we can use any pair of reference points to define Barycentric Coordinates on the line.

(a) What are the Barycentric Coordinates of $\frac{1}{4}$ if we use 2 and 5 as the reference points?

(b) What are the Barycentric Coordinates of $x = \frac{1}{4}$ if we use $-\frac{1}{4}$ and $\frac{3}{4}$ as the reference points?

Barycentric Coordinates in a Plane. Notice that it takes two coordinates to define a point on a line using Barycentric Coordinates (in fact, this is a feature of projective coordinates in general).

So to define Barycentric coordinates on a plane it should take us 3 points. To do this, instead of using two points like we did for a line, we will use 3 reference points (the vertices of a triangle).

For simplicity, we will take the approach described earlier and force our coordinates to have sum equal to 1. In this case, given any three vertices of a triangle A, B, C we will define the Barycentric Coordinates of a point x to be $(w_A : w_B : w_C)$ such that

$$x = w_A A + w_B B + w_C C$$

and $w_A + w_B + w_C = 1$.

(1) The definition given above is a bit unclear! What does it mean to multiply a point by a number and to add two points?

(a) To multiply a point $A = (x, y)$ by a number w_A results in the point $w_A A = (w_A x, w_A y)$.

(i) If $A = (1, 3)$ compute $6A$ and $\frac{1}{3}A$.

(b) To add two points $A = (x, y)$, and $B = (z, s)$ results in the point $A + B = (x + z, y + s)$.

(i) if $A = (1, 3)$ compute $6A + \frac{1}{3}A$.

(2) Show that if we use this definition on a line we get the same result as the Barycentric Coordinates we were using earlier. That is, show that if we find w_0 and w_1 such that $x = w_0 \cdot 0 + w_1 \cdot 1$ and $w_0 + w_1 = 1$ then $(w_0 : w_1)$ are exactly the Barycentric Coordinates of x .

(3) Suppose ABC is the triangle with vertices $A = (0, 1)$, $B = (-\frac{1}{2}, 0)$, and $C = (0, \frac{1}{2})$.

(a) Compute $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$, and $A + 2B - 2C$.

(b) Find the points P and Q with Barycentric Coordinates $(\frac{1}{3} : \frac{1}{3} : \frac{1}{3})$ and $(1 : 2 : -2)$ respectively.

(c) What are the Barycentric Coordinates of A ?

(d) What are the Barycentric Coordinates of the point $(-\frac{1}{4}, \frac{1}{2})$?

(4) Assume we have a fixed triangle ABC in the plane.

(a) What are the Barycentric Coordinates of A ? of B ? of C ?

(b) What are the Barycentric Coordinates of the midpoint of the line AB ?

(c) What are the Barycentric Coordinates of the midpoint of the line BC ?
What are the Barycentric Coordinates of the midpoint of the line AC ?

(d) How can you tell when a point $(w_A : w_B : w_C)$ is in the interior of the triangle?

(5) The equation of a line in Barycentric Coordinates is $xw_A + yw_B + zw_C = 0$ where x, y, z are arbitrary reals.

(a) Show that any line containing A has the form $yw_B + zw_C = 0$ from some y, z .

(b) Show that any line containing B has the form $xw_A + zw_C = 0$ for some x, z .

(c) Come up with a general equation for any line containing C .

(d) What is the equation for the line AB ? (**Hint:** This line contains both A and B , so use parts (a) and (b) of this problem!)

(e) What is the equation for the line BC ?

(f) What is the equation for the line AC ?

(g) What is the equation for the line connecting the point A to the midpoint of the line BC ?

(h) What is the equation for the line connecting the point B to the midpoint of the line AC ?

(i) What is the equation for the line connecting the point C to the midpoint of the line AB ?

- (6) Prove the following theorem: *Given any triangle ABC the three medians all intersect at a single point inside the triangle.* In addition, find the Barycentric Coordinates of this point. (**Hint:** You have already done most of the work in problem 3)

Ceva's theorem.

- (1) (Ceva's Theorem) A line is called a **cevian** if it connects a vertex of a triangle to a point on the opposite side. *Let AD , BE , and CF be cevians of the triangle ABC . Then the cevians intersect at a single point within the triangle if and only if*

$$\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = 1$$

- (a) Prove problem 4 on the previous page using Ceva's theorem.

- (b) Prove that the point D has coordinates $(0 : d : 1 - d)$ for some d . (Similarly E has coordinates $(1 - e : 0 : e)$ and F has coordinates $(f : 1 - f : 0)$ for some e, f .)

- (c) Find the equations of the lines AD , BE , and CF .

- (d) Find four equations which, if solved, imply the cevians all intersect at a single point. (it turns out that this system of equations has solutions if and only if the condition in Ceva's theorem above holds – but we will not prove this)

Homework.

- (1) Suppose m is the midpoint of the segment $[a, b]$. If we use a and b as reference points, what are the Barycentric Coordinates of m ?
- (2) Suppose we have a triangle ABC where $A = (0, 1)$ $B = (-1/2, 0)$ and $C = (0, 1/2)$. What is the equation of the line AB ? What is the equation of the line containing the points $\frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$ and $A + 2B - 2C$? (**Hint:** What are the Barycentric Coordinates of these points?)