Quadratic Equations II and Vieta's Theorem

LA Math Circle | Advanced Group

April 12, 2014

DIRECTIONS. For your solutions to these problems, you may only rely on basic algebra (and facts like $x^2 \geq 0$ for all x) or on problems that you have already solved. You may not, for example, rely on a theorem you found in a book, or something like the quadratic formula (unless you prove it in the course of your solution)

1. Show that the equation $ax^2+bx+c=0$ has real solutions if and only if the discriminant $D = b^2 - 4ac$ satisfies $D \geq 0$. Show that in this case the solutions are given by $x_{1,2} = \frac{-b\pm\sqrt{D}}{2a}$.

2. If $x_1$and x_2 are the two solutions to $ax^2 + bx + c = 0$, show

   (a) that $x_1 x_2 = \frac{c}{a}$ and $x_1 + x_2 = -\frac{b}{a}$.
   You saw a theorem in the previous handout that looked a lot like this. What was it? What's the difference and why are they different?

   (b) that the distance from x_1 to x_2 is given by $\frac{\sqrt{D}}{a}$.

3. Let $a > 0$, and let $x = ax^2 + bx + c$. Show that there exists some $R > 0$ such that $y > 0$ whenever $|x| > R$.

4. Without solving the equation $ax^2 + bx + c = 0$, where $a \neq 0$ and $D > 0$, find the sum of the squares of its roots.

5. Using TWO different methods, (i) directly using your results from Problem 2 (Vieta's Theorem), and (ii) your knowledge of roots of quadratic equations and how they relate to factors, write down a quadratic equation with integer coefficients such that its roots are equal to

   (a) $\frac{1}{2}$ and $\frac{3}{7}$.

(b) $\frac{3}{4}$ and $\frac{5}{6}$.

(c) $p + \sqrt{q}$ and $p - \sqrt{q}$. Are there any restrictions you must place on p and q?

6. Let $x_1$and $x_2$be the roots of the quadratic equation

$$x^2 - 13x - 17 = 0.$$

Write down a quadratic equation whose roots are $2-x_1$and $2-x_2$, WITHOUT finding x_1 and $x_2$explicitly.
(Hint: Use primes to denote variables in the second equation; for example, a', b', c', x'_1, x'_2 etc.)
Don't forget to check your answer.