Circuits and Paths

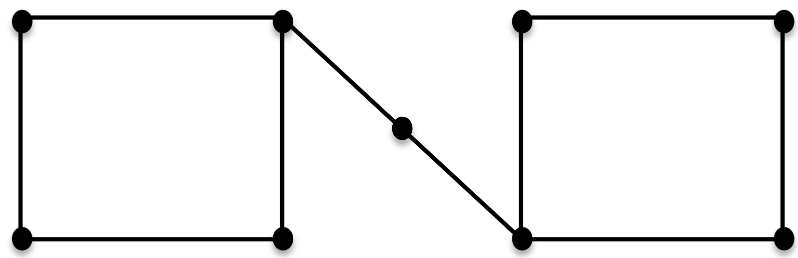
April 13, 2014

Warm Up Problem

Quandroland is an insect country that has four cities. Draw all possible ways tunnels can join the cities in Quadroland. (Remember that some cities might not be connected to each other).

Circuits and Paths

Last week, we talked about "insect worlds", which consist of cities connected by tunnels. Below is an example of an insect world.



Recall that an *even* city has an even number of tunnels connected to it, and an *odd* city has an odd number of tunnels connected to them.

- Label the cities above with the number of tunnels connected to them.
- How many even cities are there?
- How many odd cities are there?

The insect worlds we have seen are all examples of *graphs* . A *graph* is a set of *vertices* (cities) and *edges* (tunnels) connecting the vertices. An *even vertex* has an even number of edges connected to it, and an *odd vertex* has an odd number of edges connected to it. The *degree* of a vertex is the number of edges connected to the vertex.

- Draw your own graph. Label each vertex with its degree.

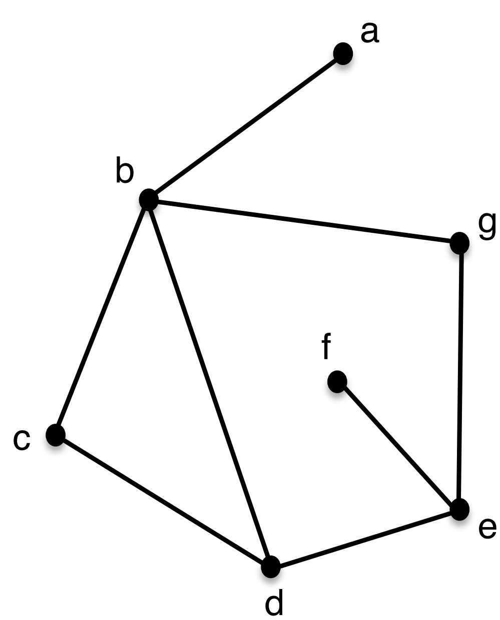
Similar to how insects can travel from one city to another city through tunnels, we can trace out a path from one vertex to another along the edges in a graph.

1. The following graph has vertices that are labeled $a-g$. Using your colored pencils, trace out the following paths on the graph below.

   (a) Using red, trace the path $d-b-c-d$

   (b) Using blue, trace the path $f-e-d-b-a$

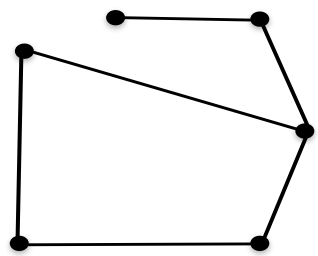
   (c) Create your own path! Use your green colored pencil to trace the path. Then, write down the path that you made.



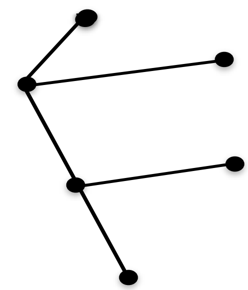
2. A *circuit* is a type of path where we start and end at the same vertex and do not use the same edge more than once. For example, the path used in part (a) in Problem 1 created a circuit.

   (a) Label which of the graphs below contain a circuit.

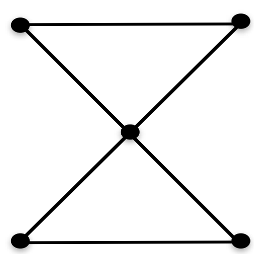
(i)



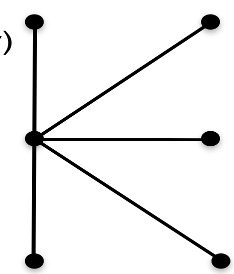
(ii)



(iii)



(iv)

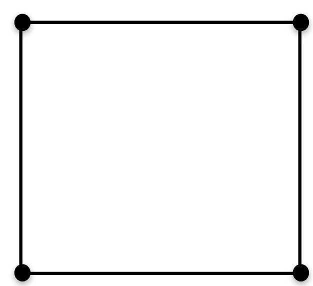


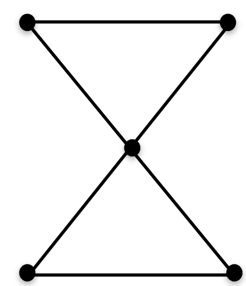
   (b) Draw your own graph that has the following properties:

      i. There is a total of six vertices and six edges.
      ii. There is a circuit made of four vertices.
      iii. There is one vertex with a degree of 3.

3. Leonhard Euler was a famous Swiss mathematician who lived in the 18^{th} century. He discovered many great mathematical properties, some involving graphs. He devised a type of circuit known as an *Euler Circuit*. An *Euler Circuit* goes through all edges in a graph, and since it is a circuit, we can only go along each edge once!

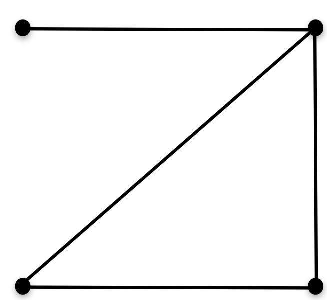
(a) The two graphs below both contain Euler Circuits. Draw your own graph with an Euler Circuit below them.



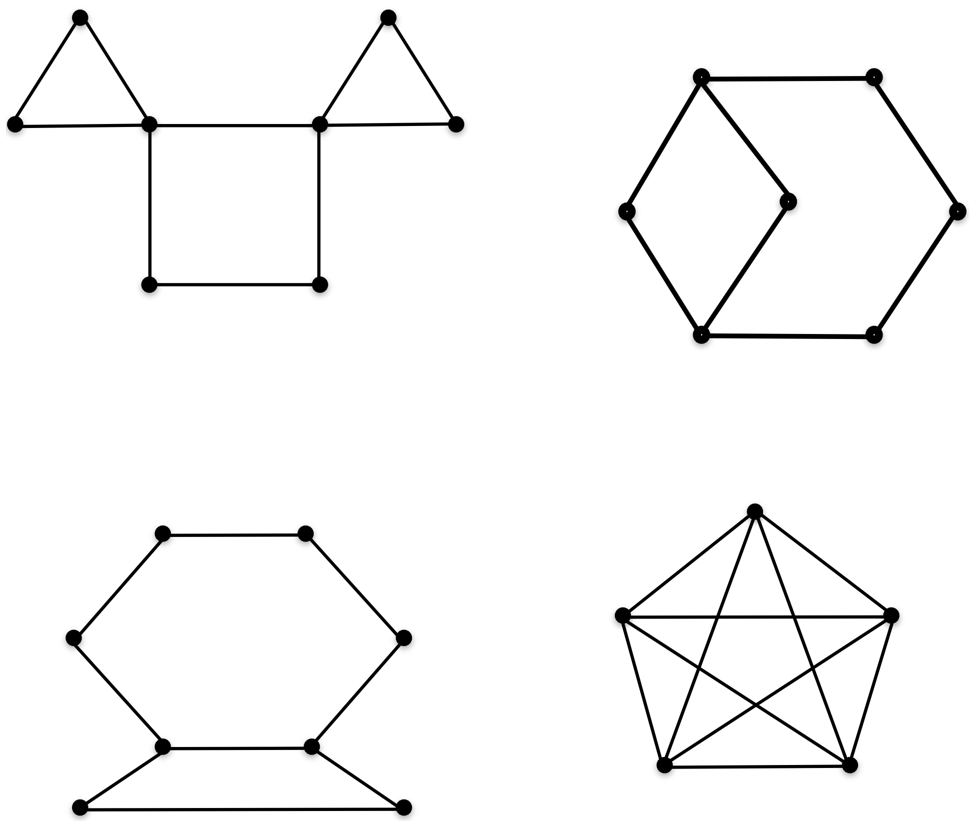


(b) (i) Does the graph below contain a circuit?

(ii) Does it contain an Euler Circuit? Explain why or why not?



4. Label which of the graphs below contain Euler Circuits.

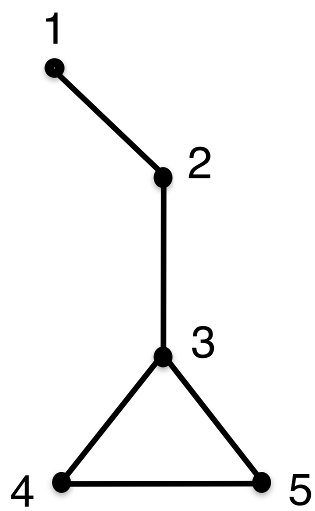


(a) For each of the graphs above, label each vertex with its degree.

(b) Do any of these graphs have only even vertices? Which ones?

(c) Using your answers in parts (b) and (d), what do you notice about graphs that have Euler Circuits?

5. Answer the questions below about the following graph.



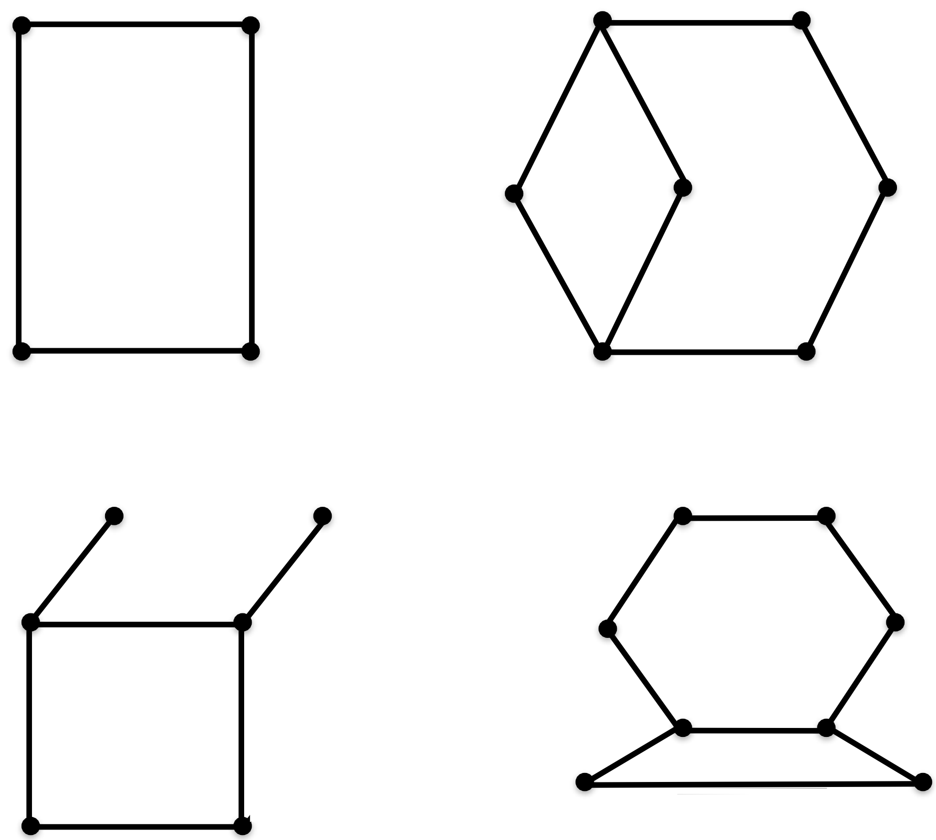
(a) If you start a path at vertex 1, are you able to trace an Euler Circuit? Explain why or why not.

(b) Change the graph by drawing an edge that connects vertex 1 with vertex 3. If you start a path at vertex 1, are you able to trace an Euler Circuit? If so, explain why adding this edge allows for an Euler Circuit.

(c) Can one decide if a given graph has an Euler Circuit by looking at the degrees of the vertices? Explain why or why not.

6. Leonhard Euler also devised a type of path known as an *Eulerian Path.* An *Eulerian Path* is a path that goes along every edge exactly once, but does not have to end at the vertex it started at.

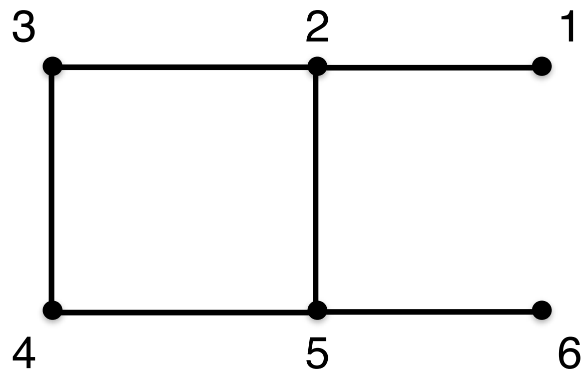
(a) In which graphs below can you make an Euler path?



(b) For each of the graphs, put a star next to the vertices that are odd.

(c) How many odd vertices can there be in graphs that have Eulerian paths?

7. Use the following graph to answer the questions below.



(a) If you start a path at vertex 1, are you able to trace an Euler path? Explain why or why not.

(b) Change the graph by drawing an edge that connects vertex 2 with vertex 6. If you start a path at vertex 1, are you able to form an Euler path? If so, explain why adding this edge allowed for an Euler path to be made.

(c) Can one decide if a given graph has an Euler path by looking at the degrees of the vertices? Explain why or why not.

8. In this problem, you will draw your own graphs and answer questions about the graphs.

   (a) Draw your own graph containing an Euler Circuit. Then, answer the following questions.

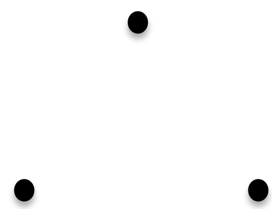
      i. Are there any odd vertices?

      ii. Is it possible to make any of the vertices odd and still have an Euler Circuit?

   (b) Draw a picture of a graph with an Euler path that has exactly two odd vertices.

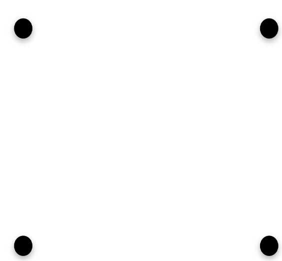
      i. Is it possible to make another vertex odd and still have an Euler path?

(c) Connect the three vertices below in such a way that there is NO Euler Circuit. Your graph must be connected, meaning that there is a path that connects any one vertex to any other vertex.



i. Can you add an edge to your graph and still not have an Euler Circuit? Explain why or why not.

(d) Create a connected graph in such a way that there is NO Euler path.



i. How many odd vertices are there in your graph?

ii. Could there be fewer odd vertices? Why or why not?