LAMC Beginners' Circle **April 13, 2014**

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Warm-up

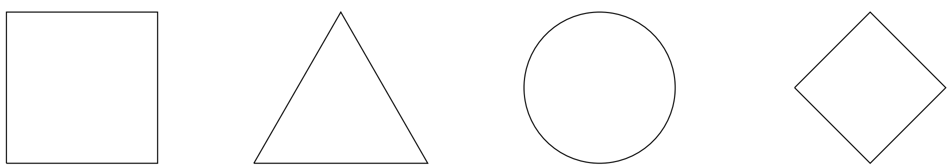
Problem 1 *Alice and Bob play the following game. Originally, the number 100 is written on the board. The players take turns subtracting either 1, or 2, or 3 from the number on the board, erasing it and writing down the result of the subtraction instead. The first player to get a negative number loses the game. Alice makes the first move. Who is going to win if both players make no mistakes?*

Back to permutations and the 15 puzzle

Problem 2 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

If you cannot do it right away, please use the following pictorial representation for the original arrangement.



$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Note that the first line of the notation we have used for writing down permutations so far is redundant. Indeed

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

means that we shuffle the second element to the first position, the first element to the second position, the fourth element to the third position, and the third element to the fourth one. Without any loss of clarity, we can write this down as

$$\sigma = \begin{pmatrix} 2 & 1 & 4 & 3 \end{pmatrix}.$$

Problem 3 *Apply the permutation* $\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$ *to the sequence of geometric figures on page 2 and draw the result in the space below.*

Problem 4 *Find the product* $\delta \circ \sigma$ *of the following two permutations.*

$$\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} \qquad \delta = \begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix}$$

$$\delta \circ \sigma = \begin{pmatrix} \qquad\qquad\qquad \end{pmatrix}$$

Let us set $\sigma^0 = e$ for any permutation σ. The permutation σ^2 is defines as $\sigma \circ \sigma$, σ^3 as $\sigma \circ \sigma^2$, and so on. Similarly, $\sigma^{-2} = \sigma^{-1} \circ \sigma^{-1}$, $\sigma^{-3} = \sigma^{-1} \circ \sigma^{-2}$, and so forth.

Problem 5 *Find the following powers of the permutation* $\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$ *from Problem 4.*

$$\sigma^2 = (\qquad\qquad\qquad)$$

$$\sigma^3 = (\qquad\qquad\qquad)$$

$$\sigma^4 = (\qquad\qquad\qquad)$$

$$\sigma^{-1} = (\qquad\qquad\qquad)$$

$$\sigma^{-2} = (\qquad\qquad\qquad)$$

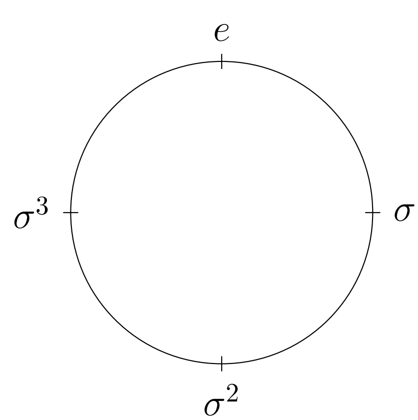
$$\sigma^{-3} = (\qquad\qquad\qquad)$$

$$\sigma^{-4} = (\qquad\qquad\qquad)$$

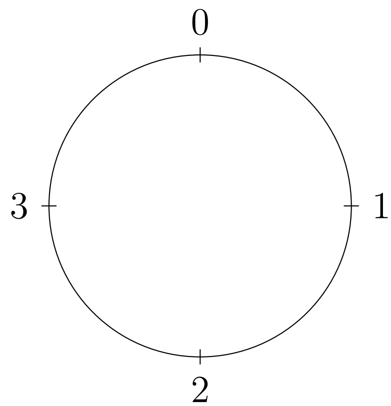
Solving Problem 5, you may have notices the following. The formula $\sigma^4 = e$ means that

- $\sigma \circ \sigma^3 = e$, hence $\sigma^{-1} = \sigma^3$ and $\sigma^{-3} = \sigma$;
- $\sigma^2 \circ \sigma^2 = e$, hence $\sigma^{-2} = \sigma^2$. Furthermore,
- $\sigma^5 = \sigma^4 \circ \sigma = e \circ \sigma = \sigma$;
- $\sigma^6 = \sigma^4 \circ \sigma^2 = e \circ \sigma^2 = \sigma^2$;
- $\sigma^7 = \sigma^4 \circ \sigma^3 = e \circ \sigma^3 = \sigma^3$;
- $\sigma^8 = \sigma^4 \circ \sigma^4 = e \circ e = e$;
- $\sigma^9 = \sigma^8 \circ \sigma = e \circ \sigma = \sigma$;
- $\sigma^{-5} = \sigma^{-4} \circ \sigma^{-1} = e^{-1} \circ \sigma^3 = e \circ \sigma^3 = \sigma^3$; and so forth.

It turns out that all the powers of the permutation σ reside naturally on the following circle.



The old-timers in this class have seen the picture before when they studied the *clock-face arithmetic*, the arithmetic of integers on a circle instead of a straight line. Let us recall. Consider the integers on a circle divided into four equal parts.



On the circle, 0 coincides with 4. We write this fact down as

$$4 \equiv 0 \ (mod\ 4) \tag{1}$$

and read it as *4 is congruent to 0 modulo 4*. The usual "=" sign is reserved for the straight number line; we use "≡" on the circle instead. The *mod* 4 symbol tells us that the circle is divided into 4 equal parts, so 4 coincides with 0, 5 with 1, 6 with 2, and so on. Or in the new notations, $4 \equiv 0 \ (mod\ 4)$, $5 \equiv 1 \ (mod\ 4)$, $6 \equiv 2 \ (mod\ 4)$, $7 \equiv 3 \ (mod\ 4)$, and so forth.

Problem 6

$$-21 \equiv \qquad\qquad (mod\ 4)$$

$$6 + 5 \equiv \qquad\qquad (mod\ 4)$$

As we can see, powers of the permutation σ from Problems 4 and 5 produce nothing more than a multiplicative realization of the *mod* 4 arithmetic. In other words, the *mod* 4 integers on the second circle serve as powers of the permutation σ on the first circle.

Example 1 *Find the 125 power of the permutation* σ *from Problems 4 and 5.*

$$\sigma^{125} = \sigma^{1\ (mod\ 4)} = \sigma$$

Problem 7 *Find the -333 power of the permutation* σ *from Problems 4 and 5.*

$$\sigma^{-333} = (\qquad\qquad\qquad\qquad)$$

The smallest positive power n of a permutation δ such that $\delta^n = e$ is called the *order of the permutation.*

Problem 8 *What is the order of the permutation* σ *we have considered in Problems 4, 5, and 7?*

Problem 9 *Find the following powers of the permutation* $\mu = (\ 3\ \ 2\ \ 4\ \ 1\ )$.

$$\mu^2 = (\qquad\qquad\qquad\qquad)$$

$$\mu^3 = (\qquad\qquad\qquad\qquad)$$

$$\mu^4 = (\qquad\qquad\qquad\qquad)$$

$$\mu^{-1} = (\qquad\qquad\qquad\qquad)$$

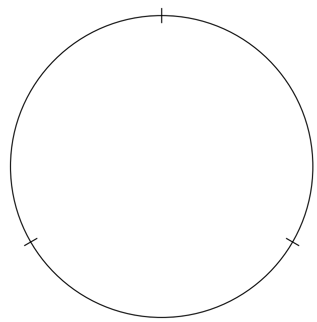
$$\mu^{-2} = (\qquad\qquad\qquad\qquad)$$

$$\mu^{-3} = (\qquad\qquad\qquad\qquad)$$

What is the order of the permutation μ*?*

The problem continues on the next page.

Mark μ^{123}, μ^{124}, and μ^{125} on the circle below.



What mod n arithmetic is realized by the powers of μ?

Further improving notations

Let us take another look at the permutation μ from Problem 9.

$$\mu = \left(3 \;\; 2 \;\; 4 \;\; 1 \right)$$

The permutation does not shuffle the second element. Hence, writing it is redundant. Knowing that the original set consists of four elements, we can write the permutation down as

$$\mu = \left(3 \;\; 4 \;\; 1 \right)$$

Since the second element does not appear in the formula, we know that the permutation does not move it. This convention becomes very convenient with larger permutations. For example, let us take another look at Sam Loyd's formulation of the

15 puzzle. Since we need to keep track of the empty square, as well as of the numbered ones, let us consider it as the 16th tile.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix}$$

switches the 14th and 15th elements only. Writing down the 14 elements it does not move is a waste of time! In the new notations,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 & 16 \end{pmatrix} = \begin{pmatrix} 15 & 14 \end{pmatrix}.$$

Since all other elements are not mentioned, we know that the permutation does not shuffle them.

Here is one more example. Let $\mu = \begin{pmatrix} 3 & 4 & 1 \end{pmatrix}$ be a permutation of six elements. Since the elements 2, 5, and 6 are not listed, μ keeps them in place. So in fact, $\mu = \begin{pmatrix} 3 & 2 & 4 & 1 & 5 & 6 \end{pmatrix}$.

Problem 10 *The permutation* $\nu = \begin{pmatrix} 3 & 1 \end{pmatrix}$ *acts on a set of three elements. Write down its full version.*

$$\nu =$$

What is the order of ν*?*

Write down the short form of $\nu^{-10{,}000{,}831}$.

$$\nu^{-10{,}000{,}831} =$$

Problem 11 *The permutation* $\delta = \begin{pmatrix} 3 & 5 & 7 & 1 \end{pmatrix}$ *acts on a set of seven elements. Write down its full version.*

$$\delta =$$

What is the order of δ*?*

Write down the short form of $\nu^{-10{,}000{,}000}$.

$$\nu^{-10{,}000{,}000} =$$

A permutation that swaps two elements and doesn't shuffle anything else is called a *transposition*. For example, the permutation that switches the order of the third and fifth element in a six-element set is $\begin{pmatrix} 5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 & 4 & 3 & 6 \end{pmatrix}$.

Problem 12 *What is the inverse of the transposition* $\begin{pmatrix} 5 & 3 \end{pmatrix}$?

$$\begin{pmatrix} 5 & 3 \end{pmatrix}^{-1} =$$

Problem 13 *What is the order of any transposition?*

Any permutation can be realized as a product of transpositions. For example, let us consider the permutation $\sigma = \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$ from Problems 4, 5, and 7. Applying the transposition $\begin{pmatrix} 2 & 1 \end{pmatrix}$ to the original order of the elements gives us the following.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 3 & 4 \end{pmatrix}$$

Let us apply the transposition $\begin{pmatrix} 4 & 1 \end{pmatrix}$ to the result.

$$\begin{pmatrix} 2 & 1 & 3 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix}$$

Finally, applying the transposition $\begin{pmatrix} 3 & 1 \end{pmatrix}$ finishes the job.

$$\begin{pmatrix} 4 & 1 & 3 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix}$$

Or more concisely,

$$\begin{pmatrix} 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \end{pmatrix}.$$

Problem 14 *Realize the permutation* $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$ *as a product of transpositions.*

Problem 15 *Write down the permutation* μ *that corresponds to the following move of the 15 puzzle. Remember, we treat the empty square as the 16th tile!*

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

$\longrightarrow$

1	2	3	4
5	6	7	8
9	10	11	
13	15	14	12

$$\mu =$$

The problem continues on the next page.

Find the product $\mu \circ \left(\begin{array}{cc} 15 & 14 \end{array}\right)$ *and compare the answer to the order of the squares on the second picture of the previous page.*

$$\mu \circ \left(\begin{array}{cc} 15 & 14 \end{array}\right) =$$

Problem 16 *Write down the permutation that corresponds to the following move of the 15 puzzle. Remember, we treat the empty square as the 16th tile!*

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	
13	**15**	**14**	**12**

$\longrightarrow$

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**		**11**
13	**15**	**14**	**12**

If you are finished doing all the above, but there still remains some time...

Recall that cryptarithmetic, also know as cryptarithm, alphametics, or word addition, is a math game of figuring out unknown numbers represented by words. Different letters correspond to different digits. Same letters correspond to same digits. The first digit of a number cannot be zero.

Problem 17 *Solve the following cryptarithm.*

$$\begin{array}{r} N\ U\ M\ B\ E\ R \\ +\ N\ U\ M\ B\ E\ R \\ \hline P\ U\ Z\ Z\ L\ E \end{array}$$

Homework

1. If you are an old-timer, please refresh your knowledge of the clock-face arithmetic. If you haven't studied it before, please learn it. The following is the link to the handout.

`http://www.math.ucla.edu/~radko/circles/lib/data/Handout-394-490.pdf`

You will find some answers and hints for the problems from the handout at the link below.

`http://www.math.ucla.edu/~radko/circles/lib/data/Handout-403-490.pdf`

2. Keep trying to figure out why Sam Loyd's configuration of the 15 puzzle is unsolvable.