Quadratic equations

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Many activities are adapted from I.M.Gelfand, A. Shen, "Algebra".
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Ancient problem: ancient and modern solution

1. Let's try to follow the reasoning in ancient Egypt to solve the following problem:

   Find two numbers whose sum is 20 and whose product is 96.

   The Egyptians argued as follows: *if the numbers would be equal, each of them would be* 10 *and the product would be* 100. *Since the product is actually* 96, *the numbers are not equal. Therefore, one of them is* $10+$ *(some number) and the other is* $10-$ *(the same number).*

   (a) Denote the number in the Egyptian argument above by x, come up with the quadratic equation that comes from the problem. Solve this equation to answer the Egyptian's question.

   (b) Solve the same problem by a more familiar method: denote one of the numbers by x and the other one by y. Write down the system of two equations encoding the conditions in the problem. Solve the system to find the answer to the problem:

   (c) Observe how the quadratic equation in the Egyptian solution (part (a)) is equivalent to the system of equations in the more familiar solution (part (b)):

The square of the sum (difference) formulas $(a \pm b)^2 = a^2 \pm 2ab + b^2$

1. Cut a square with edge $a+b$ into one square with edge a, one square with edge b and two rectangles $a \times b$ each. (You can also use algebra tiles).

2. Use algebra tiles (or a picture) to illustrate the formula

$$(a - b)^2 = a^2 - 2ab + b^2.$$

3. Compute without pencil and paper (and without calculator, of course!)

$$\begin{aligned} 101^2 &= \\ 102^2 &= \\ 99^2 &= \\ 98^2 &= \end{aligned}$$

4. What do the formulas for $(a + b)^2$ and $(a - b)^2$ give in the case $a = b$?

5. A piece of size $b \times b$ was cut from a corner an $a \times a$ square. Cut the remaining part (representing $a^2 - b^2 = (a - b)(a + b)$) into two pieces and make out of them a rectangle with sides $a - b$ and $a + b$.

Factoring polynomials

Factor the following polynomials:

1. $a^2 + 2ab + b^2$ (First, rewrite $2ab$ as $ab + ab$)

2. $a^2 + 3ab + b^2$ (First, rewrite $3ab$ as $2ab + ab$)

3. $a^2 + 3a + 2$

4. $a^2 - b^2$ (First, add and subtract ab to the expression)

5. $a^3 - b^3$ (Look where the polynomial has zero. Conclude that $a - b$ should be a factor. Then add and subtract a^2b and add and subtract ab^2)

Simplest quadratic equations

The simplest quadratic equation is of the form

$$x^2 = c, \qquad c > 0. \tag{0.0.1}$$

It can be solved as follows by factoring:

$$\begin{aligned} x^2 - (\sqrt{c})^2 &= 0 \\ (x - \sqrt{c}) \cdot (x + \sqrt{c}) &= 0. \end{aligned}$$

The last equation has two solutions: $x = \sqrt{c}$ and $x = -\sqrt{c}$, and no other solutions.

The equation $x^2 + px + q = 0$.

The main idea of solving an equation of this form is to simplify it in such a way that it becomes an equation of the simplest form (0.0.1).

Consider the following example:

$$\begin{aligned} x^2 + 2x - 6 &= 0 \\ (x^2 + 2x + 1) - 7 &= 0 \\ (x + 1)^2 &= 7 \\ (x + 1) &= \pm\sqrt{7} \\ x &= -1 \pm \sqrt{7}. \end{aligned}$$

Problem 1. Solve the following equations using the same method. Illustrate at least one of the solutions by algebra tiles or a picture.

1. $x^2 + 4x - 8 = 0$.

2. $x^2 - 2x + 2 = 0$.

In general, this method is called "completing the square" because the main step consists of adding (and subtracting) to the left side of the equation such a number $d = (p/2)^2$ that the expression $x^2 + 2 \cdot \frac{p}{2}x + d$ is a complete square, $(x + (p/2))^2$.

Problem 2. Derive the formula for the solutions of the quadratic equation

$$x^2 + px + q = 0$$

by completing the square. Be sure to consider all cases.

Vieta's theorem

Theorem 3. *(Vieta) If a quadratic equation* $x^2 + px + q = 0$ *has two different roots* x_1 *and* x_2*, then*

$$x_1 + x_2 = -p$$
$$x_1 \cdot x_2 = q$$

In other words, $x^2 + px + q = (x - x_1) \cdot (x - x_2)$*.*

Problem 4. The equation $x^2 + px + q = 0$ has roots x_1 and x_2. Find $(x_1 - x_2)^2$ in terms of p and q.

Problem 5. Find a quadratic equation with integer coefficients having $4 - \sqrt{7}$ as one of the roots.

Graphs of quadratic polynomials

How can we get the graph of a general quadratic polynomial $y = ax^2 + bx + c$ from the graph of the standard one parabola (the graph of $y = x^2$)?

Problem 6. Describe how to get the graph of the following polynomials from the graph of $y = x^2$:

1. $y = ax^2$ (consider $0 < a < 1$ and $a > 1$ separately).

2. $y = x^2 + k$.

3. $y = (x + h)^2$.

4. $y = a(x + h)^2 + k$ (Combine your results from 1-3).

Since we can always convert a quadratic polynomial into such a form by completing the square, the graph of any quadratic polynomial can be obtained from the standard parabola by vertical and horizontal shifts and stretching along the vertical axis.

Problem 7. Convert a polynomial of the form $ax^2 + bx + c$ into the form $a(x+h)^2 + k$. What is the meaning of h?

Problem 8. Determine the signs of a, b, c by looking at the graph of $y = ax^2 + bx + c$:

1. How can we determine the sign of a by looking at the graph:

2. How can we determine the sign of b by looking at the graph:

3. How can we determine the sign of c: