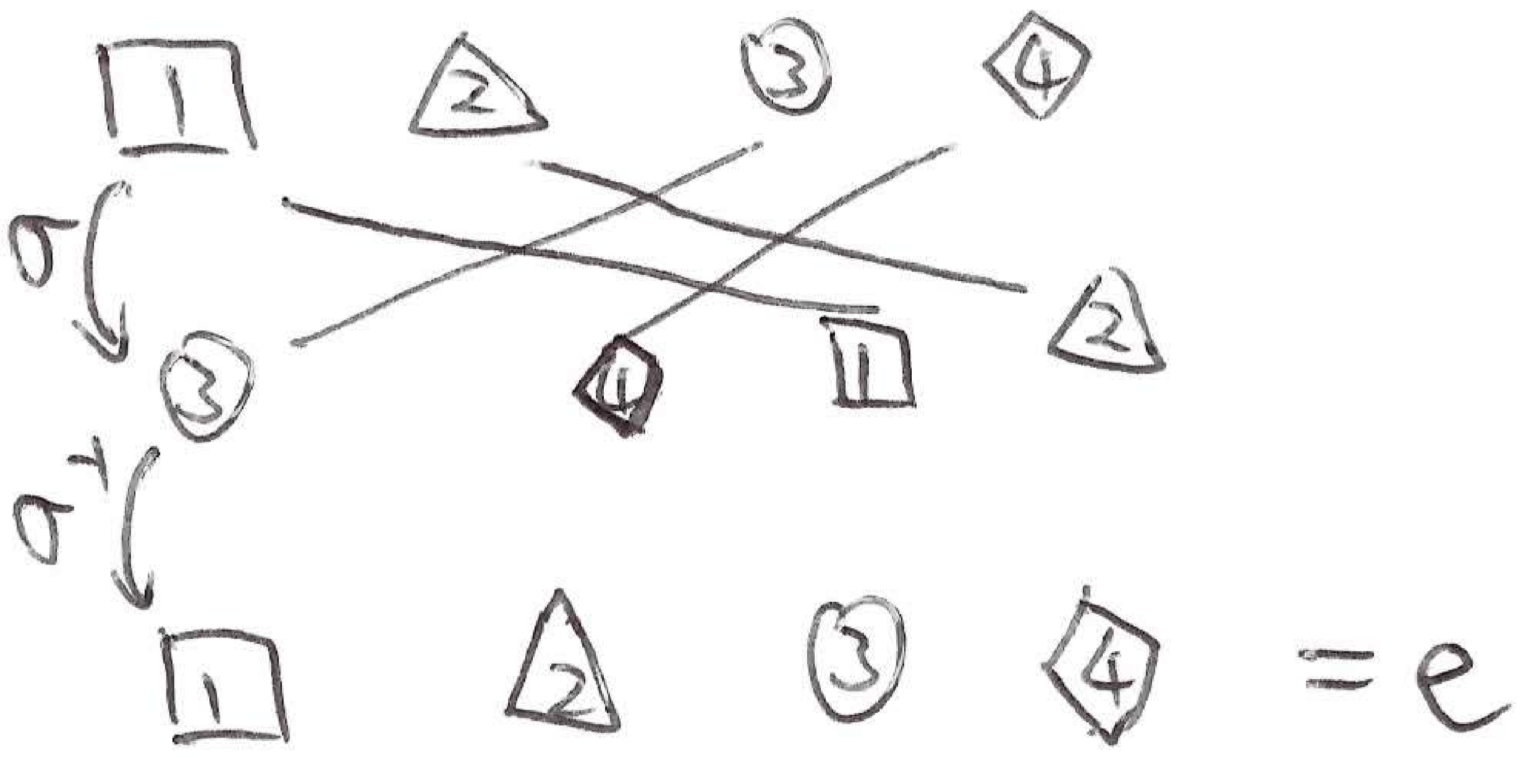
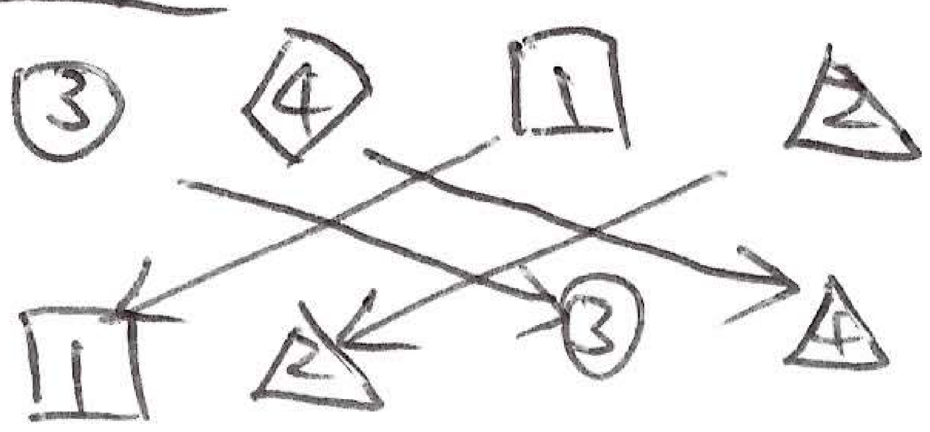


Problem 18 *Find σ^{-1} for*

Step 1:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$



Step 2

$3 \equiv 1$

$4 \equiv 2$

$1 \equiv 3$

$2 \equiv 4$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

Problems 17 and 18 exhibit two different non-trivial permutations σ that are self-inverse, $\sigma^{-1} = \sigma$. It follows from Problem 4 that there exist only two self-inverse numbers, 1 and -1, the latter being the only non-trivial. Unlike numbers, there exist lots of different non-trivial self-inverse permutations.

Problem 19 *Find a non-trivial permutation σ different from the ones in Problems 17 and 18 such that $\sigma^{-1} = \sigma$.*

An example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Please check using the definition:

$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = e$$

The 15 puzzle was invented by Noyes Palmer Chapman, a postmaster in Canastota, New York, in the mid-1870s. Sam Loyd, a prominent American chess player at the time,[1] has offered \$1,000 (about \$25,000 of modern day money) for solving the puzzle in the form shown on the picture below.

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

Proving that this particular configuration has no solution will be the primary goal of our mini-course.

Problem 20 *Write down the permutation corresponding to the Loyd's puzzle.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 \end{pmatrix}$$

[1]Ranked 15th in the world.

Sam Loyd, 1841 – 1911

Problem 21 *Let us call σ the permutation from Problem 20. Find σ^{-1}.*

$$\sigma^{-1} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14 \end{pmatrix}$$

Following the steps described before, it turns out that σ is self -inverse.

Homework

Using your copy of the 15 puzzle, attempt to solve the version suggested by Sam Loyd. Try to figure out what goes wrong.