Problem 13 *Find the permutation $\sigma \circ \delta$. If needed, use the pictorial representation as above.*

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

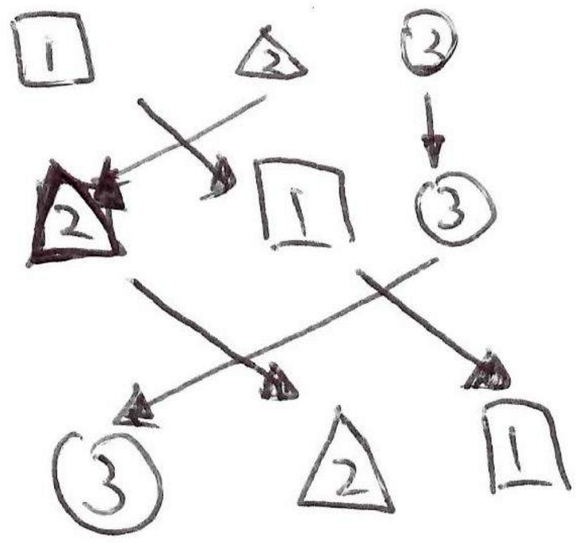
$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

δ

$$\begin{pmatrix} 2 & 1 & 3 \end{pmatrix}$$

σ

$$\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$$



<u>Note</u>: when calculating $\sigma \circ \delta$, it is important to know that you should consider the operation on the right-hand side first, in this case δ.

Also, for the second operation, it may work better if you consider numbers in its operation, i.e. $\sigma = \begin{pmatrix} \underline{1} & \underline{2} & \underline{3} \\ \underline{3} & \underline{1} & \underline{2} \end{pmatrix}$, as placeholders.

Is $\sigma \circ \delta = \delta \circ \sigma$?

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \delta \circ \sigma$$

Note that although some particular permutations may commute, multiplication of permutations in general is not a commutative operation!

Quick check:

$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \delta \circ \sigma$

Problem 14 *Find two non-trivial permutations of four elements that do commute.*

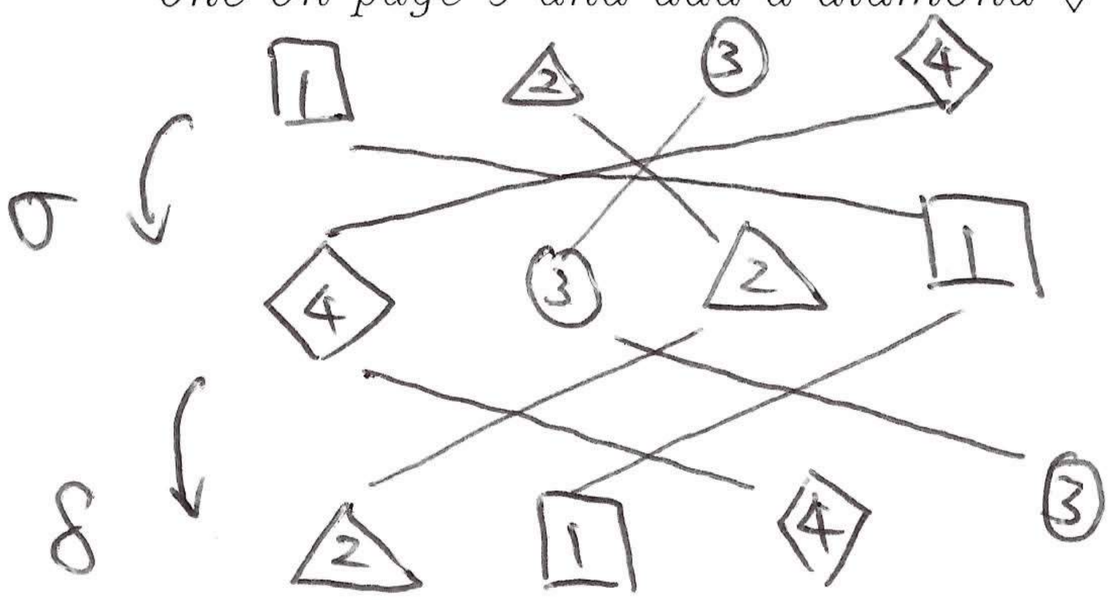
One example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

Problem 15 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

If you need to use a pictorial representation as a tool, take the one on page 5 and add a diamond $\Diamond$ as the fourth figure.

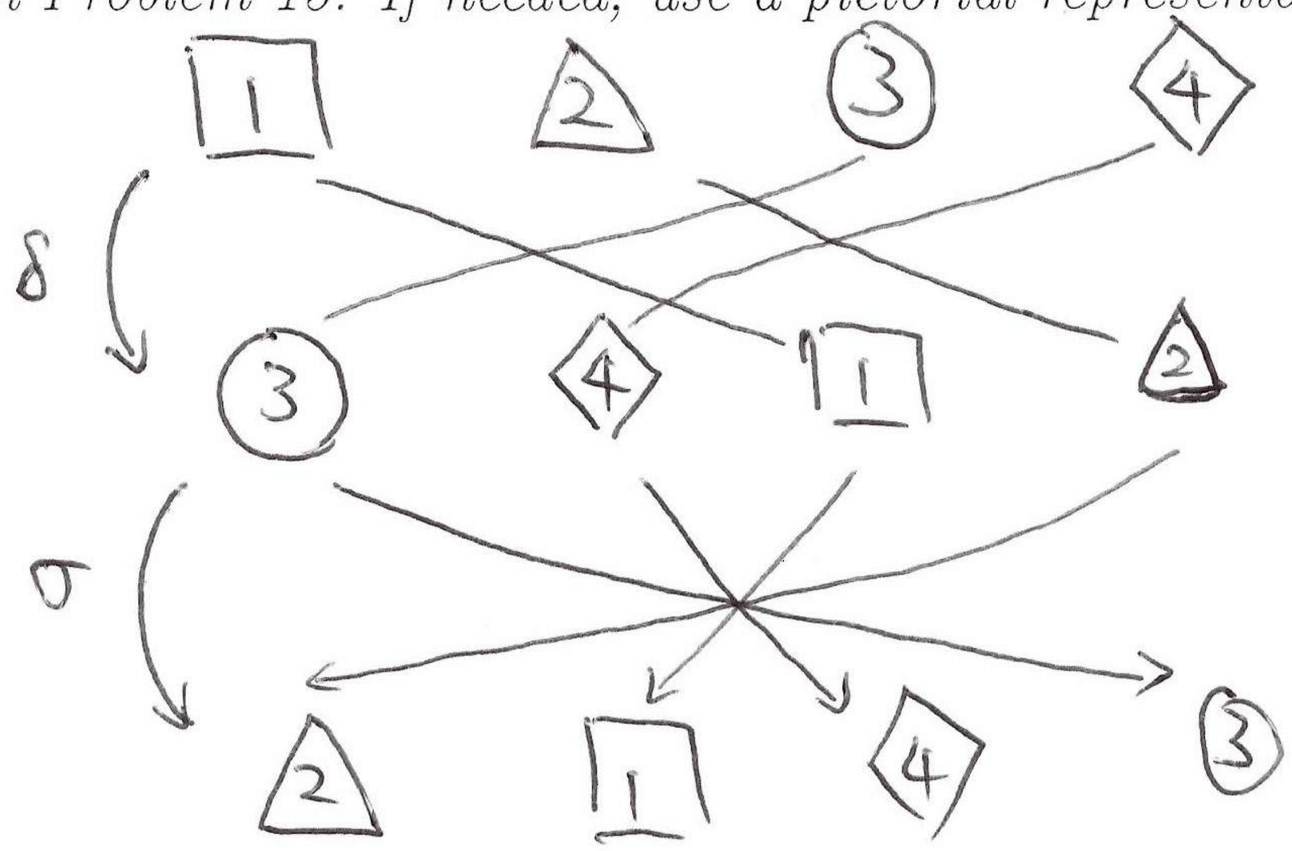


$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Problem 16 *Find the product $\sigma \circ \delta$ of the permutations*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \textit{and} \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

from Problem 15. If needed, use a pictorial representation.



$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Do the permutations δ and σ commute?

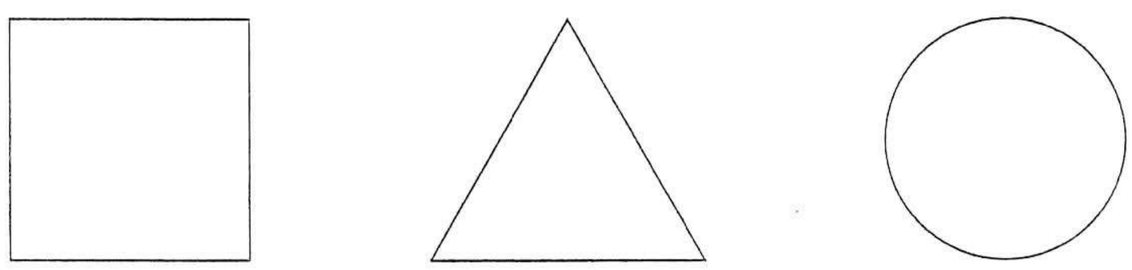
$\delta \circ \sigma = \sigma \circ \delta$

δ and σ commute.

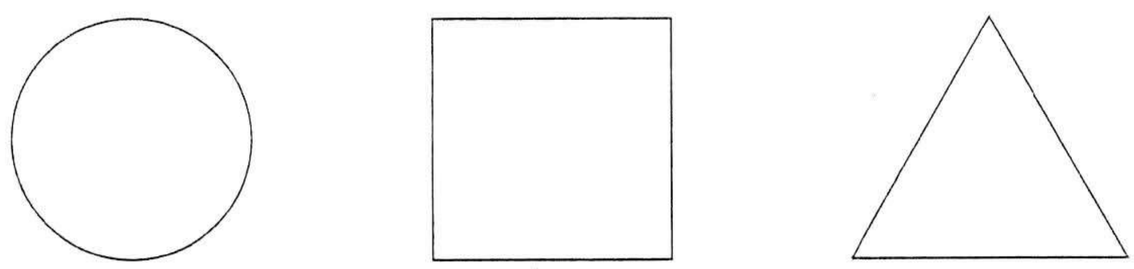
A permutation δ is called *opposite* to a permutation σ if $\delta \circ \sigma = e$. In other words, δ undoes what σ does. Such a permutation is denoted as σ^{-1} and is called the *permutation opposite to sigma* or *sigma inverse* (compare to x^{-1} on page 2).

Example 1 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

The permutation σ *reshuffles the figures*

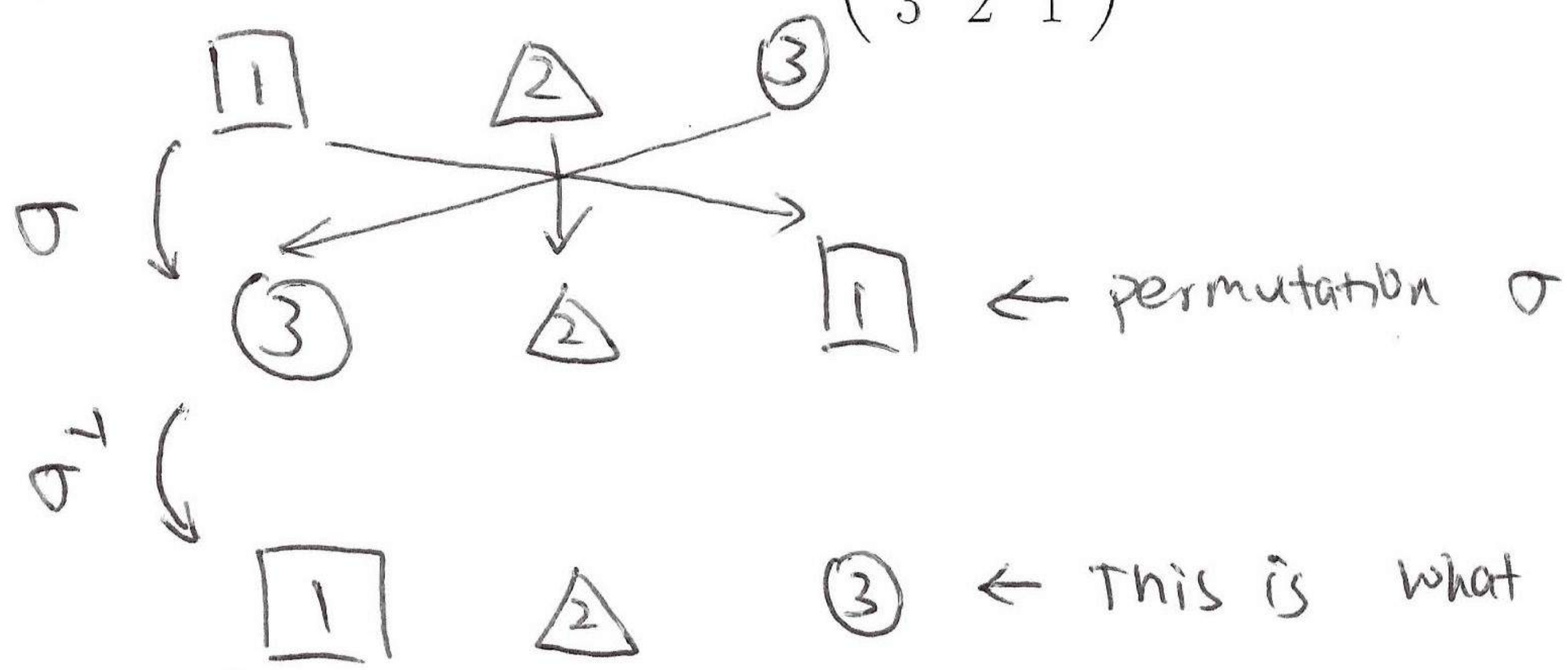


in the following order.



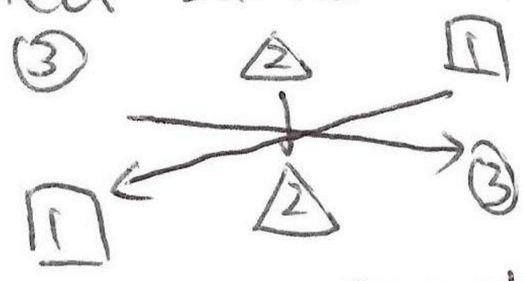
Hence, $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Problem 17 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.



Steps to find σ^{-1}:

1. connect same elements on the graph above



2. Represent this permutation by numbers/place holders

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

③ ≡ 1
△ ≡ 2
□ ≡ 3

Note that since the permutation σ^{-1} undoes what the permutation σ does, σ works the same way for σ^{-1}. Hence, not only $\sigma^{-1} \circ \sigma = e$, but $\sigma \circ \sigma^{-1} = e$ as well. Thus, σ and σ^{-1} always commute.

$$\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e \tag{1}$$