LAMC Beginners' Circle April 6, 2014

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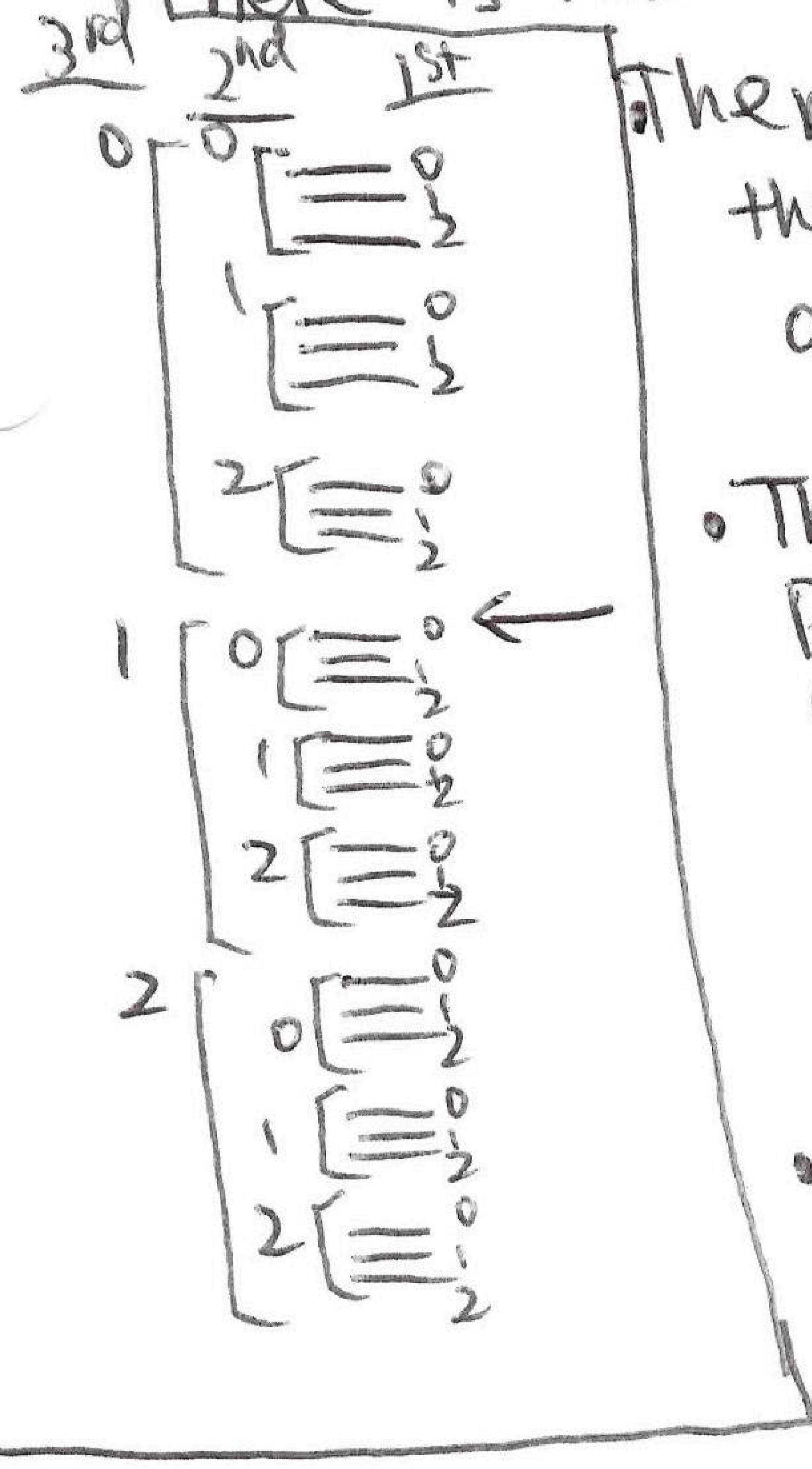
For Problem 1,
a video on youtube named "Beautiful Card Trick – Numberphile"
provides a more vivid explanation.

Warm-up

Problem 1 *Explain how the 27-card trick works.*

- When the cards went back to the pile for the last time, there is how it looks like:



There are 3 piles, each with 9 cards, when you pick them up. Denote the top pile as "0", the middle as "1", and the bottom pile as "2".

- Then, the top three cards from last top pile came from the previous top pile; the next three from the last top pile come from the previous middle pile; and the last three in the last top pile came from the previous bottom pile. Similarly, we can label the middle column by "0," "1," "2" as shown on the left.

- Also, within each group of three cards, the top one came from the top pile in the 1st round, the middle one came from the middle pile in the 1st round, and the bottom card came from the 1st bottom pile. Thus, we finish our labeling in the column on the right.

- Then, if you want the card to be the 10th from the top when you dealt the cards for the last time, then you know that you should put the pile containing that card on the top for the first time when you collected it, on the top for the second time, and in the middle for the 3rd time.

- Note that $10-1=9=1\times9+0\times3+0\times1=100_3$
  ① You need to subtract 1, because we start counting from $000_3=0$ instead of 1.
  ② You need to decode the order from lower to higher digits.
  e.g. 100_3: $0\to0\to1$; 102: $2\to0\to1$

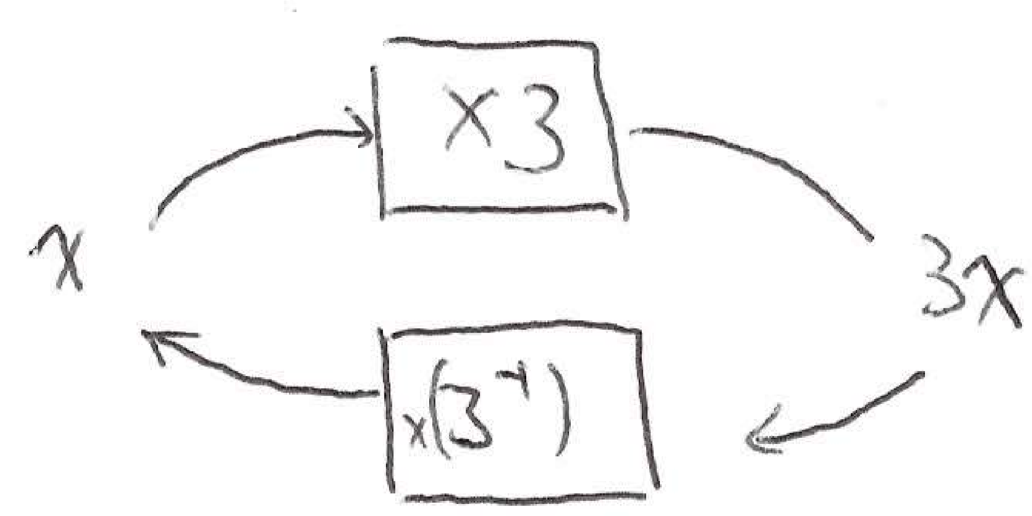
A bit of useful algebra

For a non-zero number x, the number x^{-1} (*x inverse or x to the power negative one*) is $1/x$. For example, $5^{-1} = 1/5$. The following will help us later on.

One can consider x not as a number, but as an operation of multiplying by x, $\times x$. In this case, x^{-1} becomes the operation of multiplying by x inverse, $\times 1/x$. This operation undoes what the operation $\times x$ does. For example, $78 \times 5 \times 5^{-1} = 78 \times 5 \times \frac{1}{5} = 78$.

Problem 2 *Write down the following numbers as fractions. Simplify when possible.*

$$3^{-1} = \frac{1}{3}$$



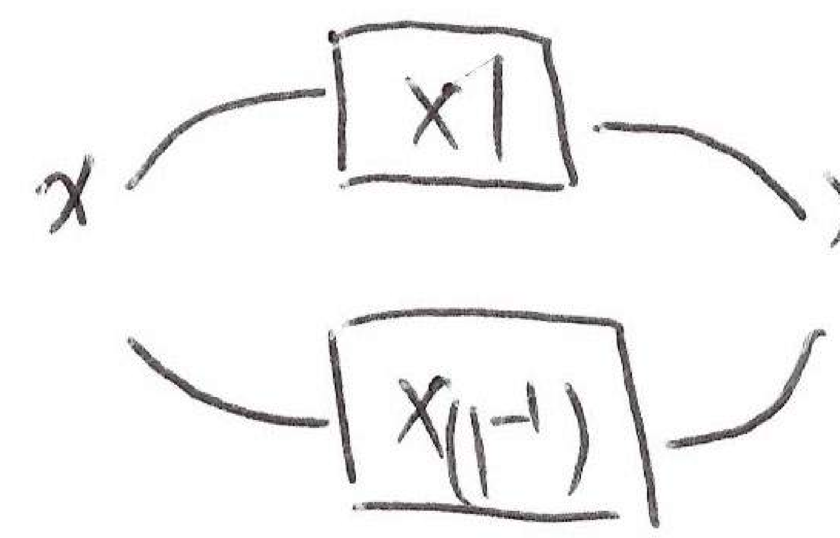
$3^{-1} = \frac{1}{3}$

$$\left(\frac{1}{3}\right)^{-1} = (3^{-1})^{-1} = 3$$

$$\left(\frac{4}{5}\right)^{-1} = \left(\left(\frac{5}{4}\right)^{-1}\right)^{-1} = \frac{5}{4} = 1.25$$

$$(0.25)^{-1} = \left(\frac{1}{4}\right)^{-1} = (4^{-1})^{-1} = 4$$

$1^{-1} = 1$



$1^{-1} = 1$

Problem 3 *Compute the following. Simplify when possible.*

$$49 \times 7^{-1} = 49 \times \frac{1}{7} = 7$$

$$4^{-1} \times 3 \times 2 = \frac{1}{4} \times 3 \times 2 = \frac{3 \times \cancel{2}}{\cancel{4}\,2} = \frac{3}{2} = 1.5$$

$$23 \times 10^{-1} = 23 \times \frac{1}{10} = \frac{23}{10} = 2\frac{3}{10} = 2.3$$

$$23 \times 10^{-2} = 23 \times \frac{1}{10^2} = \frac{23}{100} = 0.23$$

$$23 \times 10^{-3} = 23 \times \frac{1}{10^3} = \frac{23}{1000} = 0.023$$

$$23 \times 10^{-2014} = 23 \times \frac{1}{10^{2014}} = \frac{23}{10^{2014}} = 0.\underbrace{0 \cdots 0}_{2012 \text{ zeros}}23$$

Note: $10^{-x} = (10^x)^{-1} = \frac{1}{10^x}$, as $(10^x)^{-1}$ is the "inverse operation" of 10^x.

Problem 4 *Find a number x different from one such that $x^{-1} = x$. How many numbers have this property (coincide with their own inverses)?*

$$x^{-1} = x$$

$$x^{-1} \cdot x = x \cdot x$$

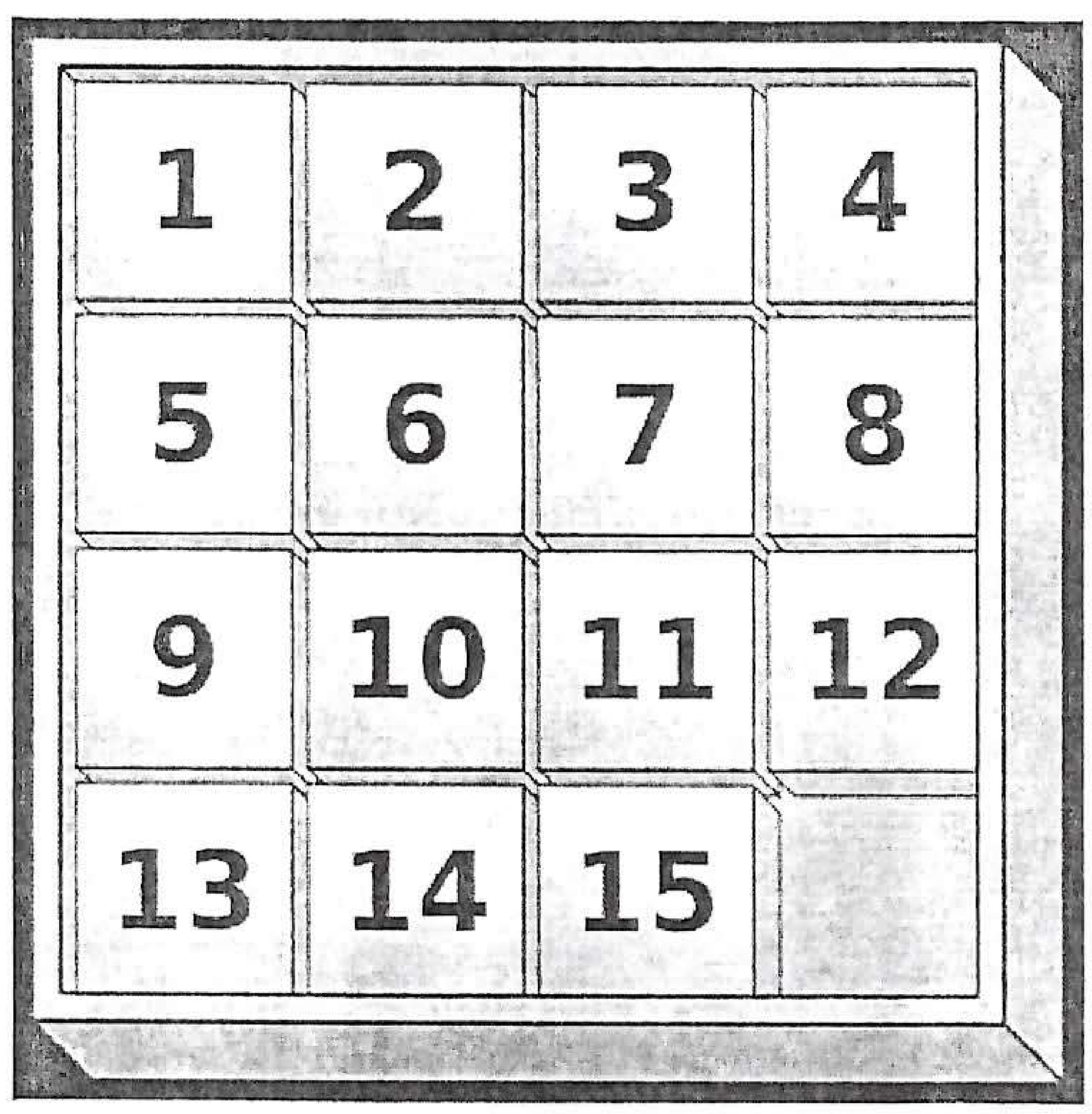
by definition $\quad 1 = x^2$

$\Rightarrow \quad x = -1$, as $x \neq 1$

Permutations and the 15 puzzle

The ultimate goal of the mini-course we begin now is to learn solving the *15 puzzle* (when a solution exists).

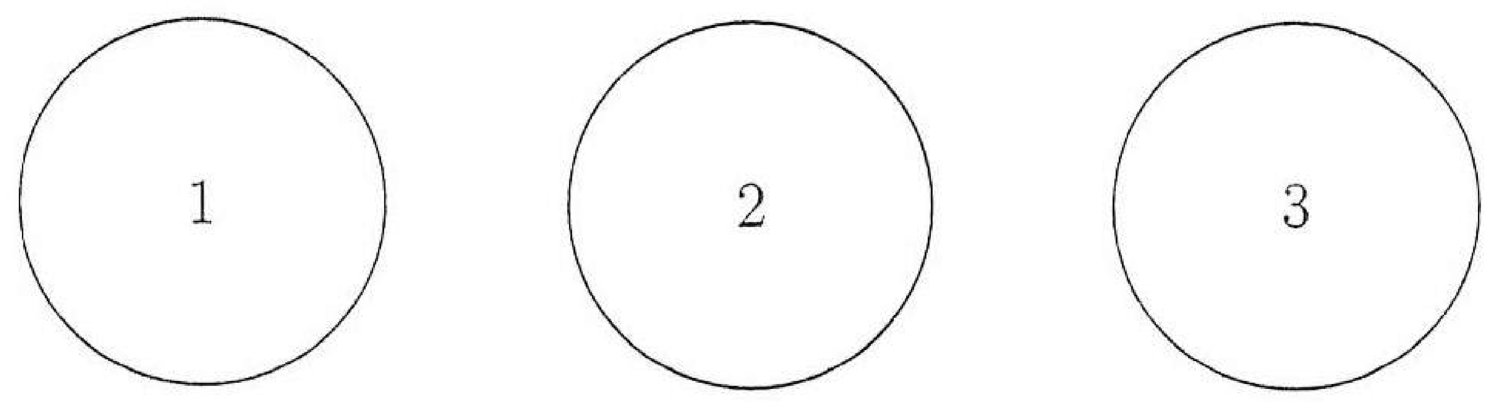
The puzzle consists of a 4×4 frame randomly filled with 15 squares numbered one through fifteen. The objective is to slide the squares in the proper order, left to right, starting with the top row as on the picture below.



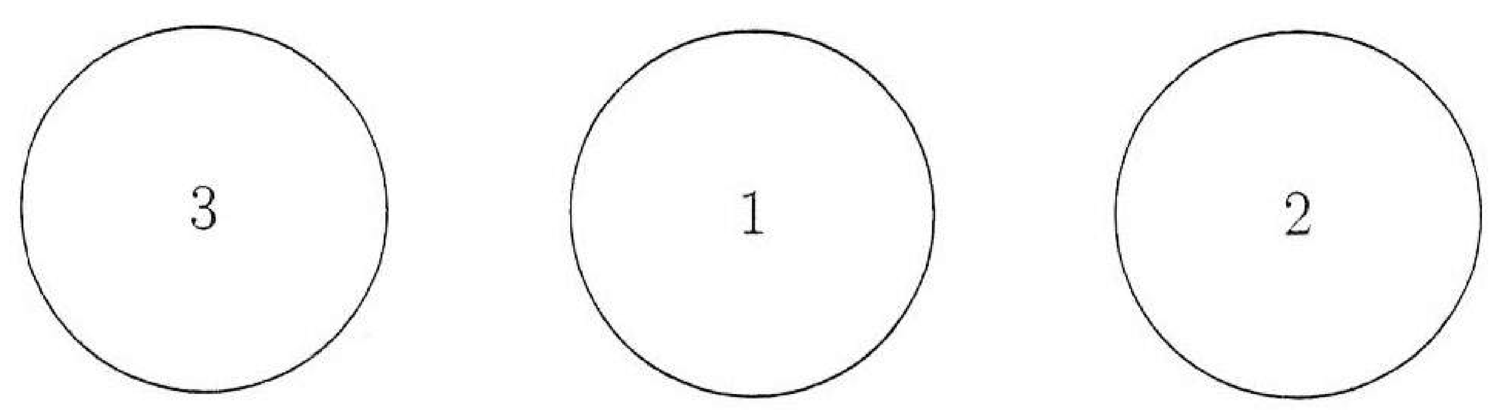
15 puzzle

The mathematical foundation of the solution is the theory of *permutations*. The theory not only helps to unravel the puzzle, but also comes quite handy in a wide variety of applications, from card tricks to quantum mechanics. It will take us a few classes to learn the basics. In the meantime, you are encouraged to get the puzzle, either in the solid form or as a smartphone/tablet app, and to start playing!

Consider a set of marbles numbered 1 through n. Originally the marbles are lined up in the order given by their numbers. The following picture shows an example with $n = 3$.



Then the marbles are reshuffled in a different order.



A *permutation* is the operation of reshuffling the marbles (or elements of any set). The one shown in the example is written down as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Instead of the numbered marbles, we can reshuffle distinguishable elements of any set. For example, let us consider the following geometric figures rather than the marbles numbered 1, 2, and 3.