LAMC Beginners' Circle **April 6, 2014**

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Warm-up

Problem 1 *Explain how the 27-card trick works.*

A bit of useful algebra

For a non-zero number x, the number x^{-1} (*x inverse or x to the power negative one*) is $1/x$. For example, $5^{-1} = 1/5$. The following will help us later on.

One can consider x not as a number, but as an operation of multiplying by x, $\times x$. In this case, x^{-1} becomes the operation of multiplying by x inverse, $\times 1/x$. This operation undoes what the operation $\times x$ does. For example, $78 \times 5 \times 5^{-1} = 78 \times 5 \times \frac{1}{5} = 78$.

Problem 2 *Write down the following numbers as fractions. Simplify when possible.*

$$3^{-1} =$$

$$\left(\frac{1}{3}\right)^{-1} =$$

$$\left(\frac{4}{5}\right)^{-1} =$$

$$(0.25)^{-1} =$$

$$1^{-1} =$$

Problem 3 *Compute the following. Simplify when possible.*

$$49 \times 7^{-1} =$$

$$4^{-1} \times 3 \times 2 =$$

$$23 \times 10^{-1} =$$

$$23 \times 10^{-2} =$$

$$23 \times 10^{-3} =$$

$$23 \times 10^{-2014} =$$

Problem 4 *Find a number x different from one such that $x^{-1} = x$. How many numbers have this property (coincide with their own inverses)?*

Permutations and the 15 puzzle

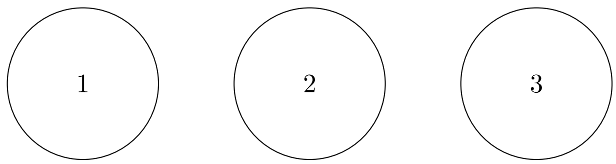
The ultimate goal of the mini-course we begin now is to learn solving the *15 puzzle* (when a solution exists).

The puzzle consists of a 4×4 frame randomly filled with 15 squares numbered one through fifteen. The objective is to slide the squares in the proper order, left to right, starting with the top row as on the picture below.

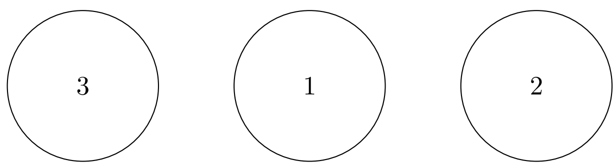
15 puzzle

The mathematical foundation of the solution is the theory of *permutations*. The theory not only helps to unravel the puzzle, but also comes quite handy in a wide variety of applications, from card tricks to quantum mechanics. It will take us a few classes to learn the basics. In the meantime, you are encouraged to get the puzzle, either in the solid form or as a smartphone/tablet app, and to start playing!

Consider a set of marbles numbered 1 through n. Originally the marbles are lined up in the order given by their numbers. The following picture shows an example with $n = 3$.



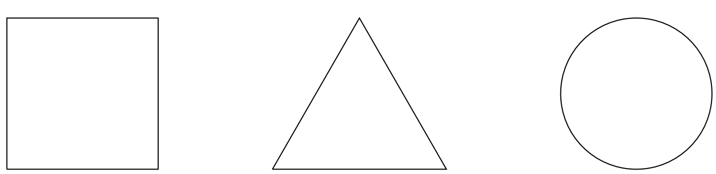
Then the marbles are reshuffled in a different order.



A *permutation* is the operation of reshuffling the marbles (or elements of any set). The one shown in the example is written down as follows.

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Instead of the numbered marbles, we can reshuffle distinguishable elements of any set. For example, let us consider the following geometric figures rather than the marbles numbered 1, 2, and 3.

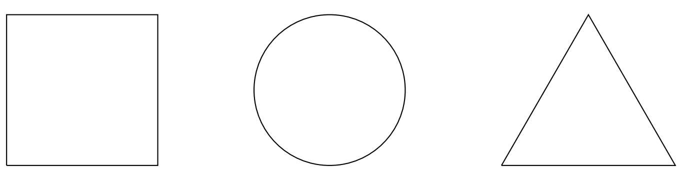


Then the permutation

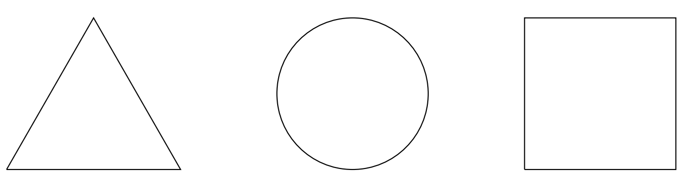
$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

will reshuffle the figures into the following order.

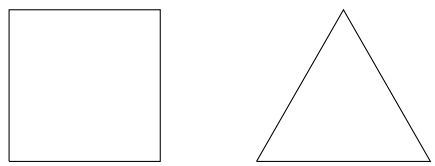
Problem 5 *For the original order of figures given on page 5, write down the permutations that correspond to the following pictures.*



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$



$$\begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that the last permutation does not reshuffle anything at all. Permutations of this kind typically denoted as e and called *trivial*. A trivial permutation is still a permutation, and an important one!

Problem 6 *Write down the trivial permutation for $n = 5$.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ & & & & \end{pmatrix}$$

Problem 7 *For the original order of figures given on page 5, draw the figures in the orders prescribed by the permutations below. Use the space to the right of a permutation to draw the corresponding picture.*

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

Recall that $n! = 1 \times 2 \times \ldots \times (n-1) \times n$. For example, $3! = 1 \times 2 \times 3 = 6$.

Problem 8 *Compute* $5!$

$$5! =$$

Problem 9 *How many permutation of four elements are there?*

Problem 10 *How many permutation of* $n+1$ *elements are there?*

Problem 11 *Write down a permutation of four elements.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 12 *Write down a permutation of four elements that keeps the third element in place.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Find the number of permutations of four elements that keep the third element in place.

The number =

It is possible to combine, or *multiply*, permutations. For example, let us apply the permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

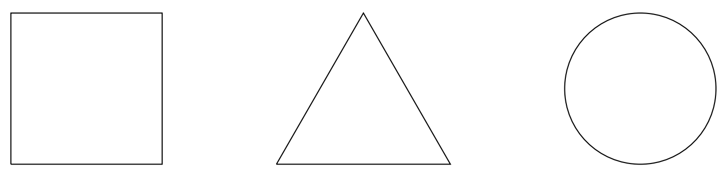
to the marbles already reshuffled by the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The permutation δ switches the first and second elements, so

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

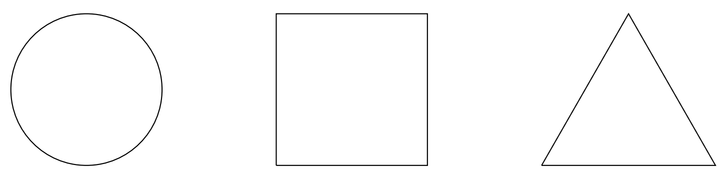
Let us take another look at the above computation using the figures from page 5. Originally, the set of the figures is ordered as follows.



The permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

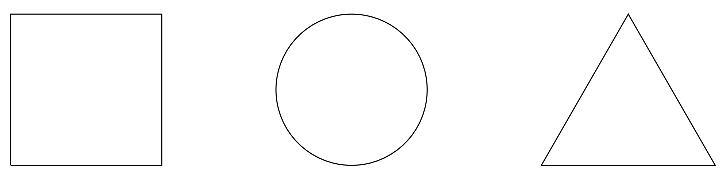
produces the picture below.



The permutation

$$\delta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

applied to the latter configuration gives us the following.



Comparing the last picture to the original gives us the answer.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Note that in the product $\delta \circ \sigma$ of permutations, it is the one on the right, σ, that acts first on the set it permutes!

Problem 13 *Find the permutation $\sigma \circ \delta$. If needed, use the pictorial representation as above.*

$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Is $\sigma \circ \delta = \delta \circ \sigma$?

Note that although some particular permutations may commute, multiplication of permutations in general is not a commutative operation!

Problem 14 *Find two non-trivial permutations of four elements that do commute.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 15 *Find the product $\delta \circ \sigma$ of the following two permutations.*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \qquad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

If you need to use a pictorial representation as a tool, take the one on page 5 and add a diamond $\Diamond$ as the fourth figure.

$$\delta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problem 16 *Find the product $\sigma \circ \delta$ of the permutations*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

from Problem 15. If needed, use a pictorial representation.

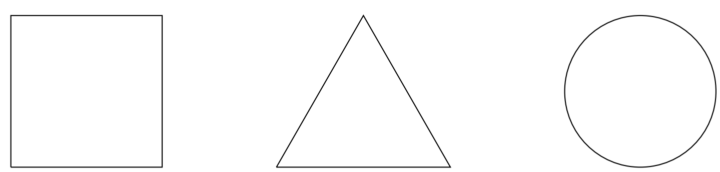
$$\sigma \circ \delta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Do the permutations δ and σ commute?

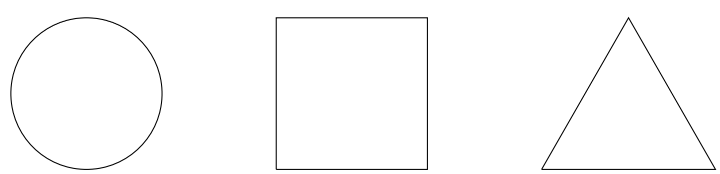
A permutation δ is called *opposite* to a permutation σ if $\delta \circ \sigma = e$. In other words, δ undoes what σ does. Such a permutation is denoted as σ^{-1} and is called the *permutation opposite to sigma* or *sigma inverse* (compare to x^{-1} on page 2).

Example 1 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

The permutation σ *reshuffles the figures*



in the following order.



Hence, $\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

Problem 17 *Find* σ^{-1} *for* $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$.

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ & & \end{pmatrix}$$

Note that since the permutation σ^{-1} undoes what the permutation σ does, σ works the same way for σ^{-1}. Hence, not only $\sigma^{-1} \circ \sigma = e$, but $\sigma \circ \sigma^{-1} = e$ as well. Thus, σ and σ^{-1} always commute.

$$\sigma^{-1} \circ \sigma = \sigma \circ \sigma^{-1} = e \tag{1}$$

Problem 18 *Find* σ^{-1} *for*

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ & & & \end{pmatrix}$$

Problems 17 and 18 exhibit two different non-trivial permutations σ that are self-inverse, $\sigma^{-1} = \sigma$. It follows from Problem 4 that there exist only two self-inverse numbers, 1 and -1, the latter being the only non-trivial. Unlike numbers, there exist lots of different non-trivial self-inverse permutations.

Problem 19 *Find a non-trivial permutation* σ *different from the ones in Problems 17 and 18 such that* $\sigma^{-1} = \sigma$.

The 15 puzzle was invented by Noyes Palmer Chapman, a postmaster in Canastota, New York, in the mid-1870s. Sam Loyd, a prominent American chess player at the time,[1] has offered $1,000 (about $25,000 of modern day money) for solving the puzzle in the form shown on the picture below.

1	**2**	**3**	**4**
5	**6**	**7**	**8**
9	**10**	**11**	**12**
13	**15**	**14**	

Proving that this particular configuration has no solution will be the primary goal of our mini-course.

Problem 20 *Write down the permutation corresponding to the Loyd's puzzle.*

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ & & & & & & & & & & & & & & \end{pmatrix}$$

[1]Ranked 15th in the world.

Sam Loyd, 1841 – 1911

Problem 21 *Let us call σ the permutation from Problem 20. Find σ^{-1}.*

$$\sigma^{-1} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ & & & & & & & & & & & & & & \end{pmatrix}$$

Homework

Using your copy of the 15 puzzle, attempt to solve the version suggested by Sam Loyd. Try to figure out what goes wrong.