

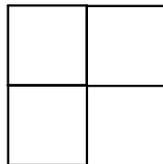
## ON THE CHESS BOARD

LAMC INTERMEDIATE - 3/09/14

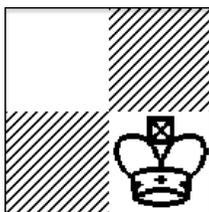
This week our problems derive from the chess board. However, they are not all chess related problems.

### A BRIEF INTRODUCTION TO INDUCTION

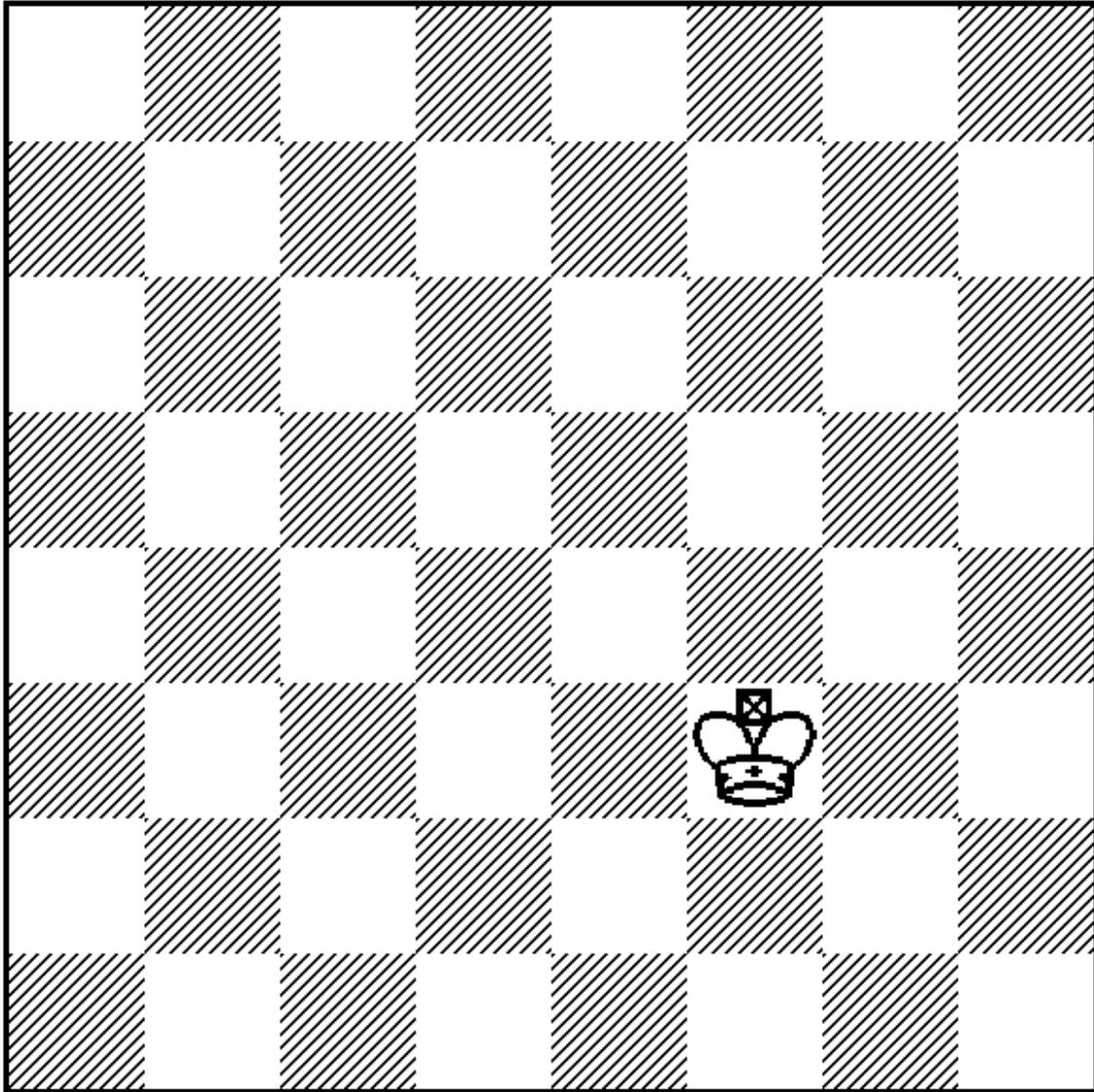
Consider the following situation. Morgan has an unusual chess board of size  $2^n \times 2^n$  (If  $n = 3$  this board is not actually that unusual!). Derek removes one square from the board (by placing a king on it) and challenges Morgan to fill up the rest of the board with triominoes. A triomino is the following piece:



- (1) If  $n = 1$ , Morgan has a  $2 \times 2$  board. This time Morgan immediately sees where to place his triomino. Indicate where Morgan will place his triomino.



- (2) Okay, that wasn't too hard. But things can get a lot more complicated very quickly! Suppose we increase  $n$  to 3 (What are the dimensions of the board in this case?). Derek removes a random square and Morgan does not see any obvious way to fill the board. Find a way to fill the board in the figure below.



(3) From the previous exercise we see that things will get very complicated, and we have lost hope of proving that Morgan can always cover the board by giving a simple strategy. Our only hope now is **induction**!

- Induction works as follows. Suppose you have infinitely many statements you want to prove  $p_1, p_2, \dots$ . If you know  $p_1$  is true and that  $p_n$  implies  $p_{n+1}$  then you can conclude that  $p_k$  is true for all  $k$ .

– Explain why this should be true:

(a) We want to prove that Morgan can always find a way to fill his board. What are our statements  $p_1, p_2, \dots$ ?

(b) Is  $p_1$  true?

(c) Does  $p_n$  imply  $p_{n+1}$ ? (Draw a picture to explain!)

**Homework: Induction and Another Chess Board Tiling.** The following problems are to be considered homework. The next hand-out requires students to know how chess pieces move. If you already understand how they move very well, you can work on these homework problems while the instructor at your table helps others with the warm-up from that hand-out.

(1) We want to prove that  $1 + 2 + \cdots + n = \frac{1}{2}n(n + 1)$  for all  $n$ .

(a) What are  $p_1, p_2, \dots$ ?

(b) Is  $p_1$  true? (Prove it)

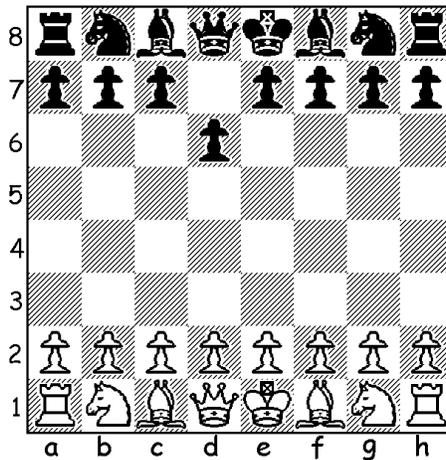
(c) Does  $p_n$  imply  $p_{n+1}$ ? (Prove it)

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- (2) Prove by induction that  $\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{n}{n+1} = \frac{1}{n+1}$ .
- (3) Suppose Derek has removed two opposite corners from a standard chess board. Can Morgan cover the board with **dominoes**?
- (4) (**Challenge**) A round-robin tournament is a tournament in which every team plays every other team. A cycle is a sequence  $T_1 > T_2 > \cdots > T_n > T_1$  where  $T > L$  represents team  $T$  beat team  $L$ . Show that every round-robin tournament which contains a cycle contains a cycle of length three (Note: this is the smallest possible cycle  $T_1 > T_2 > T_3 > T_1$ ).

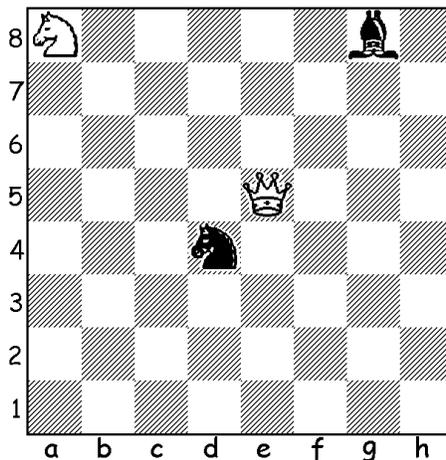
## PROBLEMS WITH CHESS PIECES ON UNUSUAL BOARDS

**Warm-Up:** Problems in this section require you to know how chess pieces can move. Talk with the instructor at your table about how they move and see if you can solve the following problems to get a feel for moving the pieces.

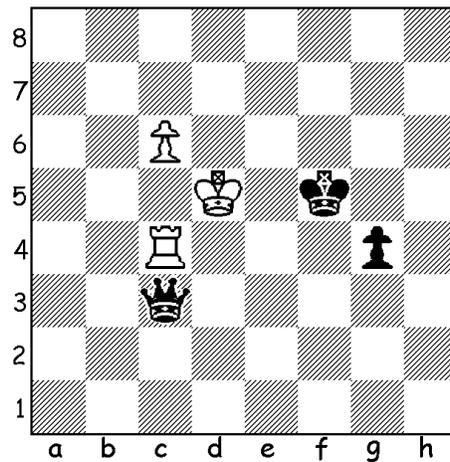
- (1) (Courtesy of Kate) Is the following a legal chess position?



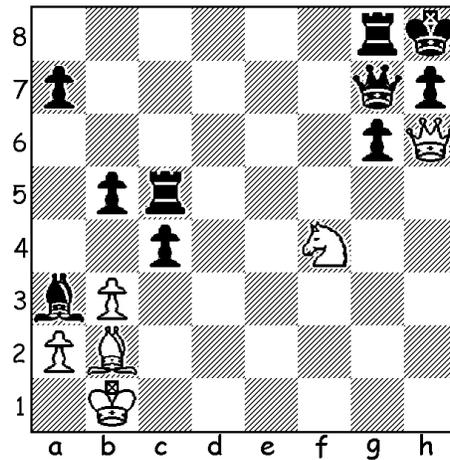
- (2) To which positions can each of the pieces on the following diagram move?  
(Note: This is clearly not a legal chess position! There are no kings!)



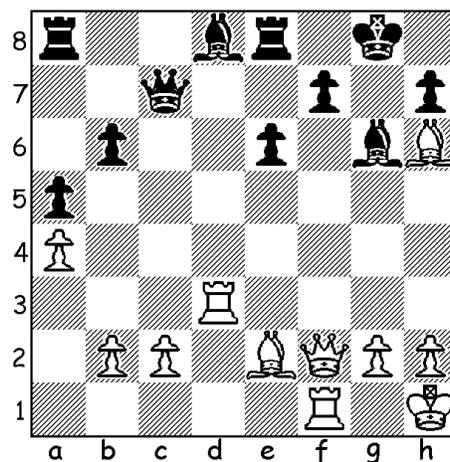
(3) (Keres-Fischer 1959) Black to move.



(4) White to move.

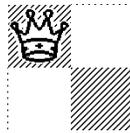


(5) This one is **much** harder than the previous ones (and is just here for fun).  
White to move.



**Dominating numbers.** On a given board, the dominating number of a piece is the minimum number of copies of this piece we can place on the board so that every square is either occupied or attacked.

For example, on a  $2 \times 2$  board the Queen Dominating number is 1 by placing the queen as follows:



- (1) What is the Rook dominating number on the  $2 \times 2$  board above? What about the King dominating number?

- (2) What is the Rook dominating number on a  $1 \times n$  board?

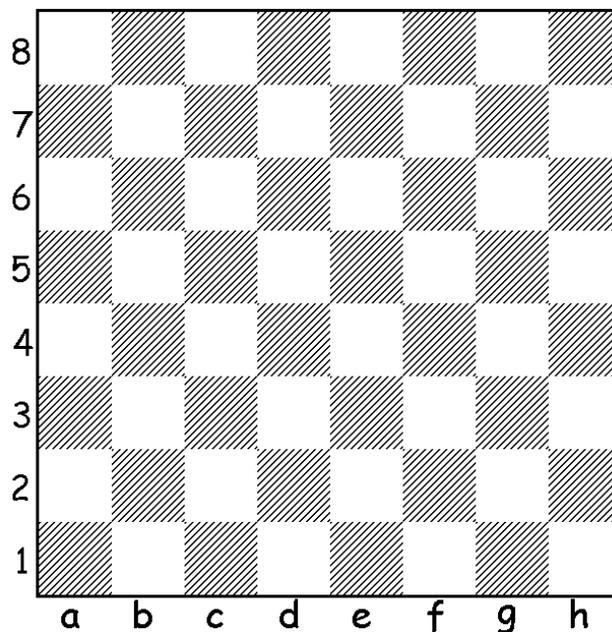
- (3) What is the Rook dominating number on an  $n \times n$  board?

(4) Let's take a closer look at the Queen dominating number of square boards.

(a) What is the Queen dominating number of a  $3 \times 3$  board?

(b) What is the Queen dominating number of a  $4 \times 4$  board?

(c) In general it is quite difficult to find the Queen dominating number. It turns out that the dominating number on the standard  $8 \times 8$  board is 5. Find a way to place 5 queens to dominate the following  $8 \times 8$  board.



**Independence.** Pieces on the chess board are called independent if none of them can attack any other. An interesting question is: “What is the maximum number of independent \_\_\_\_\_ we can place on the chess board?”

For example, we can place exactly two independent rooks on the  $2 \times 5$  chess board as follows:



(1) What is the maximum number of independent knights we can place on a  $2 \times 2$  board?

(2) Can you place four independent bishops on a  $3 \times 3$  board?

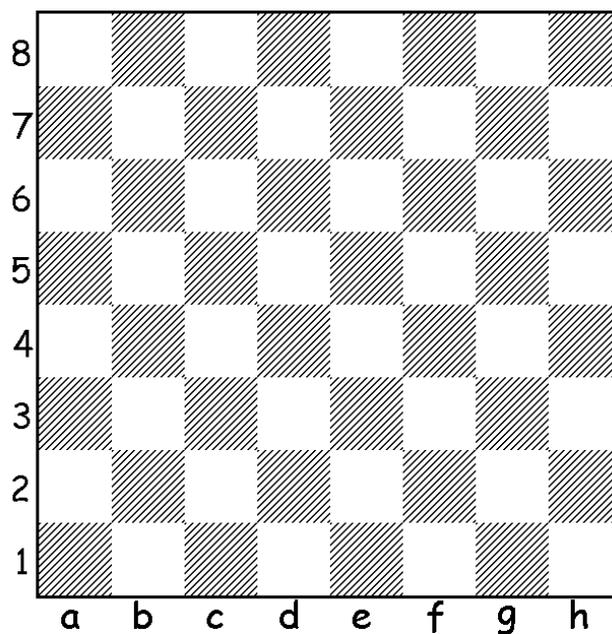
(3) Can you place eight independent rooks on an  $8 \times 8$  board?

(4) Let's look at Queens again.

(a) What is the maximum number of independent queens you can place on a  $2 \times 2$  board?

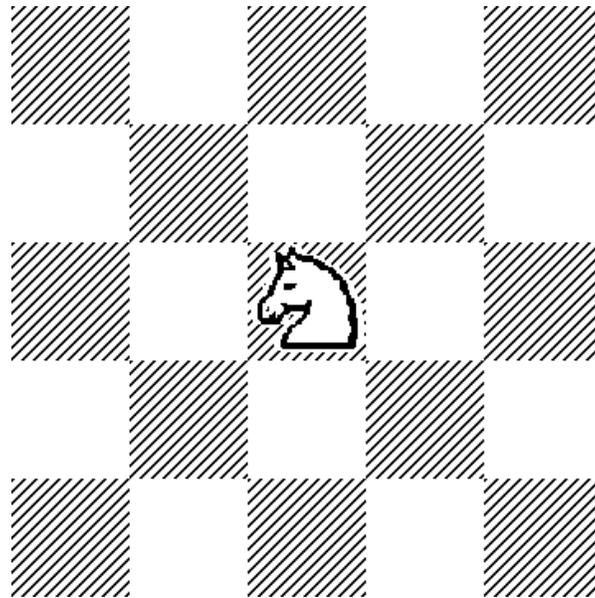
(b) What is the maximum number of independent queens you can place on a  $3 \times 3$  board?

(c) Surprisingly enough (based off your previous results) for  $n \neq 2, 3$  we can actually place  $n$  independent queens on the board. Place 8 independent queens on the following chess board.



**More stuff...** It turns out there are many other interesting questions we can ask about chess boards, and you should even try to come up with your own questions and answer them!

If you have finished all the other problems on this handout, you should try the following challenge. Can you move the knight so that it visits every square exactly once?



Note: This can also be done on an  $8 \times 8$  board and many others... If you manage to draw a knights tour on this  $5 \times 5$  board, try to do it on the  $8 \times 8$  board. If you find this fun, I recommend you visit the [Wikipedia article on knights tours](#) for an interesting read.