MATH KANGAROO (LEVEL 7-8)

LAMC INTERMEDIATE - 3/02/14

All problems come from old Math Kangaroo tests, however, the possible answers have been removed (to deter you from guessing). Solve the problems in any order, they are not in order of difficulty.

(1) Find the sum $x + y$ if x and y fulfill the conditions of the following equation:

$$(x - y - 1)^2 + (x + y - 7)^2 = 0$$

Answer: The answer is 7. This follows because both summands are non-negative so must be 0, hence $x + y - 7 = 0$.

(2) A square piece of paper with dimensions $10cm \times 10cm$ has been cut into squares with area $25cm^2$ each. Then each square is cut into two triangles. How many triangles are there?

Answer: There are 8 triangles. The big square has area $100cm^2$ so there are $100/25 = 4$ little squares. Each makes two triangles giving us 8 in total.

(3) The numbers $1, 2, 3, \cdots, 1022, 1023, 1024$ have been placed around a circle clockwise in the order given. We move around the circle clockwise and erase every other number until only one number is left. Which number will be the last one left if we erase number 1 first.

Answer: The answer is 1024. This follows because on the k-th time around the circle, we erase odd multiples of 2^k. Since $1024 = 2^{10}$ it will be the last number erased.

(4) A square of a positive number is 500% greater than that number. What is the number?

Answer: The answer is 6. The given information says $x^2 = x + 5x$ so x had better be 6.

(5) What is the greatest number of elements that can be chosen from $\{1, 2, 3, 4, \cdots, 25\}$ so that the sum of any two of them is not divisble by three? (Give a maximal such set.)

Answer: The answer is 10. We rewrite the numbers as $3 \cdot 0 + 1, 3 \cdot 0 + 2, 3 \cdot 1, \cdots, 3 \cdot 8 + 1$. Then if we choose $\{3k + 1 : k = 0, \cdots, 8\}$ the sum of any two of these is $3l + 2$ for some l and hence is not divisible by three. Then we can also add in any single nultiply of 3 and we're still good, giving us 10 total. Notice that if we have more than 10, then we would have to select either two multiples of three (which is no good) or $3j + 1$ and $3m + 2$ but the sum of these is $3(j + m) + (2 + 1) = 3(j + m + 1)$ a multiple of three.

(6) A certain father is 52 years old, and his sons are 24 and 18 years old. How many years later will the age of the father be the same as the sum of the ages of both his sons?

Answer: The answer is 10 years. We simply need to solve $52 + x = (24 + x) + (18 + x) = 42 + 2x$ that comes out to $10 = x$. Without algebra, one could say the difference is currently 10 years, and each year the difference decreases by 1 so it will take a total of 10 years.

(7) All positive whole number which are equal to the product of their proper factors (factors which are not 1 or the whole number) are written in ascending order. What is the sixth number that will be written?

Answer: The sixth one is 21. We just list them, $6, 8, 10, 14, 15, 21, \cdots$

(8) What is the ones digit of the product of all the primes less than 2014?

Answer: The answer is 0. Both 2 and 5 are primes, so 10 divides this product.

(9) How many pairs of digits a, b from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ make the following equation true: $a \cdot b = 10 + a$.

Answer: The answer is 2. We cannot take $a = 9, 8, 7, 6, 4,$ or 3 because for each of these $10 + a$ is not a multiple of a. Then we also cannot take $a = 1$ or 0 since $b < 10$. So a has to be 2 or 5. For each choice of a there is a unique solution b.

(10) There are seven cards in a box. Numbers from 1 to 7 are written on these cards, exactly one number on each card. The first sage takes 3 cards at random from the box and the second sage takes 2 cards. 2 cards are left in the box. Then, the first sage says to the second one: "I know the sum of the numbers of your cards is even." The sum of the numbers on the cards of the first sage is equal what? (Can you figure out the first sage's exact cards?)

Answer: 12. The sage knows that the sum of any two of the remaining four cards is even. For this to be the case all the remaining cards must have the same parity. This seems like two possible cases, luckily for us, there are only three evens, so all the remaining cards must be odd. Hence the sage has all the evens ($2 + 4 + 6 = 12$).

(11) The number a is positive and less than 1, and the number b is greater than 1. Which of the following numbers is greatest?

$$a \times b, \quad \frac{b}{a}, \quad \frac{a}{b}, \quad b, \quad b - a$$

Answer: The answer $\frac{b}{a}$. Since $a < 1$ we know that $\frac{1}{a} > 1$ and so $\frac{b}{a} > b$ and $b > b \times a$ since $a < 1$ and also $b > b - a$ trivially and $b > \frac{a}{b}$ since $b > 1$ and $a < 1$. Thus the largest is $\frac{b}{a}$.

(12) Dustin chooses a 5-digit positive integer and deletes one of its digits to make a 4-digit number. The sum of this 4-digit number and the original 5-digit number is 52713. What is the sum of the digits of the original 5-digit number?

Answer: Suppose he deleted the first digit, then we have $abcde+bcde = 52713$ but this is impossible because this says $e+e = 3$ or $e+e = 13$ etc. all impossible. Other cases are similarly impossible, so we must deleted the last digit. Giving us

$$abcde + abcd = 52713$$

This tells us that we must have $a = 4$, (if a were 5 we would have to carry in the second last addition giving us 6 in the first digit of the sum). By arguing in the exact same way, we conclude that b cannot be 8 since we would have to carry in the next column, forcing us to have $b = 7$. Then we are quickly forced to have $c = 9$ so $d = 2$ and finally $e = 1$. Thus our original number was actually

$$47921$$

with digit sum $4 + 7 + 9 + 2 + 1 = 23$

(13) A certain boy always tells the truth on Thursdays and Fridays, always lies on Tuesdays, and randomly tells the truth or lies on other days of the week. On seven consecutive days he was asked what his name was, and on the first six days he gave the following answers in this order: Pax, Morgan, Pax, Morgan, Deven, Morgan. What did he answer on the seventh day?

Answer: Pax. He tells the truth on consecutive days, and he hasn't said the same answer consecutively yet. But if he said Morgan it would force the day to be Friday. And three days earlier on Tuesday he said his name was Morgan, a contradiction. Therefore he must say Pax.

(14) The positive integers x, y, z satisfy $x \cdot y = 14, y \cdot z = 10,$ and $x \cdot z = 35$. What is $x + y + z$?

Answer: The answer is 14. y must divide both 14 and 10 so it can either be 2 or 1. If it is 1 then $x = 14$ and $z = 10$ so $xz = 140$ a contradiction. Thus $y = 2$ but then we can just solve agian to get $x = 7$ and $z = 5$. So the total sum is 14.

(15) The British mathematician August de Morgan claimed that he was x years old in the year of x^2. He is known to have died in 1899. When was he born?

Answer: 1806. We know that $1849 = 43^2$.

(16) The product of my children's ages is equal to 1664. The oldest child is twice as old as the youngest. How many children do I have. (What are their ages?)

Answer: 3. At first glance this seems tough to answer. One way to do it is to factor 1664 into primes. $1664 = 2^7 * 13$. Since 13 only appears once, the oldest and youngest children must have ages which are powers of two. This immediately implies that the oldest has to have age > 13 as well, and hence at least 2^4. The only way this can happen is if his/her age is exactly 2^4 (since the youngest child would then need to be 2^3). Giving us exactly three children ages $8, 13, 16$.

(17) When 999 was divided by a certain two digit number the remainder was 3. What is the remainder when 2001 is divided by this number?

Answer: 9. Again this seems tough at first glance, but we can factor $996 = 83 * 12$ so we see the only possible candidates for this two digit number are 83 and 12. Modulo either one of those we get 9.

(18) A pair of integers is called *good* if their sum is equal to their product. How many good paris of integers are there?

Answer: The answer is 2. The pairs are $(0,0)$ and $(2,2)$. We can immediately see that the relation can't work if any number has absolute value bigger than 2 and we also can't have any number be 1 or -1. So you can simply test all possible pairs in this region and get these as the answer.

(19) What is the first digit of the smallest number in which the sum of the digits equals 2014?

Answer: 7. We want all trailing 9's so we can have the number as short as possible. $9 \cdot 223$ is 2007 leaving us with 7 more to get 2014.

(20) A bottle of a volume of $\frac{1}{3}$ liter is filled $\frac{3}{4}$ with juice. How much juice will be left in the bottle after pouring out $\frac{1}{5}$ of a liter?

Answer: $\frac{1}{20}$ of a liter. This follows because $\frac{1}{4} - \frac{1}{5} = \frac{1}{4 \cdot 5}$.

(21) What is the value of the expression:

$$(1^2 + 2^2 + 3^2 + \cdots + 2014^2) - (1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + 2013 \cdot 2015)?$$

Answer: The answer is 2014. This is because $(a-1)(a+1) = a^2 - 1$ so each term on the right is 1 smaller than the corresponding term on the left and there are 2014 terms (for this reasoning you need to count $0 \cdot 2$ as a term on the right, even though it's not written).

(22) The sum of the smallest three-digit number whose digits add up to 8 and the largest three-digit number whose digits add up to 8 is?

Answer: The answer is 907. The smallest is 107 and the largest is 800.

(23) An island is inhabited by two types of people: truth-tellers and liars. The truth-tellers always speak the truth and the liars always lie. 25 of the island's inhabitants stood in a line. Each of them with the exception of the first person said: *The person directly in front of me is a liar,* while the person standing first in the line said: *Everyone standing behind me is a liar.* How many liars stood in the line?

Answer: The answer is 13. We look at the people in order. The first person cannot be telling the truth, since then the third person would also be telling the truth a contradiction. Thus the second person is telling the truth. Now by uniformity of their statements we see that every other person is a liar. Hence there are liars at every odd position, giving 13 liars.

(24) How many ten digit numbers can be constructed using the digits $1, 2$ and 3 so that any two consecutive digits differ by exactly one?

Answer: The answer is $64 = 2^6$. Every other digit has to be a two. This means we can have a two in positions $2, 4, 6, 8, 10$ or a two in positions $1, 3, 5, 7, 9$. In either case we have 2^5 ways to arrange the other numbers. Giving us a total of $2^5 \cdot 2 = 2^6$.

(25) Consider pairs of positive integers with sum no larger than 103 and quotient smaller than $\frac{1}{3}$. What is the largest possible quotient of any such pair?

Answer: The answer is $\frac{25}{76}$. If we list our pairs as (a, b) we must have $3a < b$ and $a + b \leq 103$. Since $3a < b$ and we want the quotient as large as possible we should look at pairs of the form, $(a, 3a + 1)$ the largest such a will then be the best for us. The largest a we can choose is 25 by the first condition in the problem, giving us the result.

(26) Each of 18 cards is numbered with either a 4 or a 5. It turns out that the sum of all the numbers is divisible by 17. How many cards are labeled with a 4?

Answer: The answer is 5. We look at which multiple of 17 we got. It must be at least $5 \cdot 17$ because $4 \cdot 17$ is too small (even if we had all 4s our total sum would be $4 \cdot 18$). But $6 \cdot 17$ is too large (it's bigger than $5 \cdot 18$). So we must have $5 \cdot 17$ as our total sum. But this is only possible if there are 5 fours!

(27) Derek, who is an avid fisherman, caught 12 fish over three consecutive days. On each day, he caught more fish than on the previous day. On the third day he caught fewer fish than the total number from the previous two days. How many fish did Derek catch on the third day?

Answer: He caught 5 fish. The total of the first two days has be more than 6. But because of this he must have caught fewer than 5 fish on the second day. Thus the only possibility if 4 fish the second day and 3 the first day giving us 7 in the first two days, or 5 on the third day.

(28) What is the angle formed by the hour and minute hand of a clock at $4:40$ pm?

Answer: 100 degrees. (It is tempting to say 120 degrees, but remember, at $4:40$ the hour hand isn't pointing at the 4!). There are 30 degrees between the two nearest numbers on the clock. We find the location of the hour hand at $4:40$. It is $2/3$ of the way between the four and the five. So it is located at 140 degrees (measured from the top of the clock). The minute hand is exactly at the eight, which is at $30*8=240$ degrees. The difference is 100 degrees, as claimed.

(29) How many solutions that can be expressed with positive integers does the equation below have?

$$a^2b+14=2014$$

Answer: There are 6 solutions. Factor $2000=2^4\cdot 5^3$ then we just need to count how many squares divide this. By inspection, the following are all the possibilities for a: $1,2,5,2^2,5\cdot 2,5\cdot 2^2$ giving us 5.

(30) If $\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{a}{b}=9$, then what is $a+b$?

Answer: The answer is 35. The whole product is $\frac{a}{2}$ so $a=18$ and $b=17$ giving us a sum of 35.