LAMC Beginners' Circle **March 2, 2014**

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Warm-up

Problem 1 *Six cars are driving along the same highway from the city A to the city B. The distance all the six cars have covered together is 75 miles. The total distance all the cars still have to cover is 45 miles. How long is the way from A to B?*

Problem 2 *Ten straight lines are drawn through a common point in the Euclidean plane. Prove that at least one of the angles they form is less than* 20°.

Use the inclusion-exclusion principle to solve the following two problems.

Problem 3 *Find all the integers from 1 to 100 that are not divisible by 2 or 3.*

Problem 4 *Find all the integers from 1 to 100 that are not divisible by 2, 3, or 5.*

Find a way to solve the following two problems that

1. is easy to generalize and

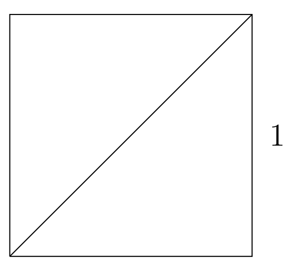
2. does not use the inclusion-exclusion principle.

Problem 5 *How many numbers from 0 to 99 contain the digit 3?*

Problem 6 *How many numbers from 0 to 999 contain the digit 3?*

Some of our students have already solved the following problem suggested by Ethan Kogan. However, Ethan's solution is very beautiful and different from what we have seen, so let us discuss the problem one more time.

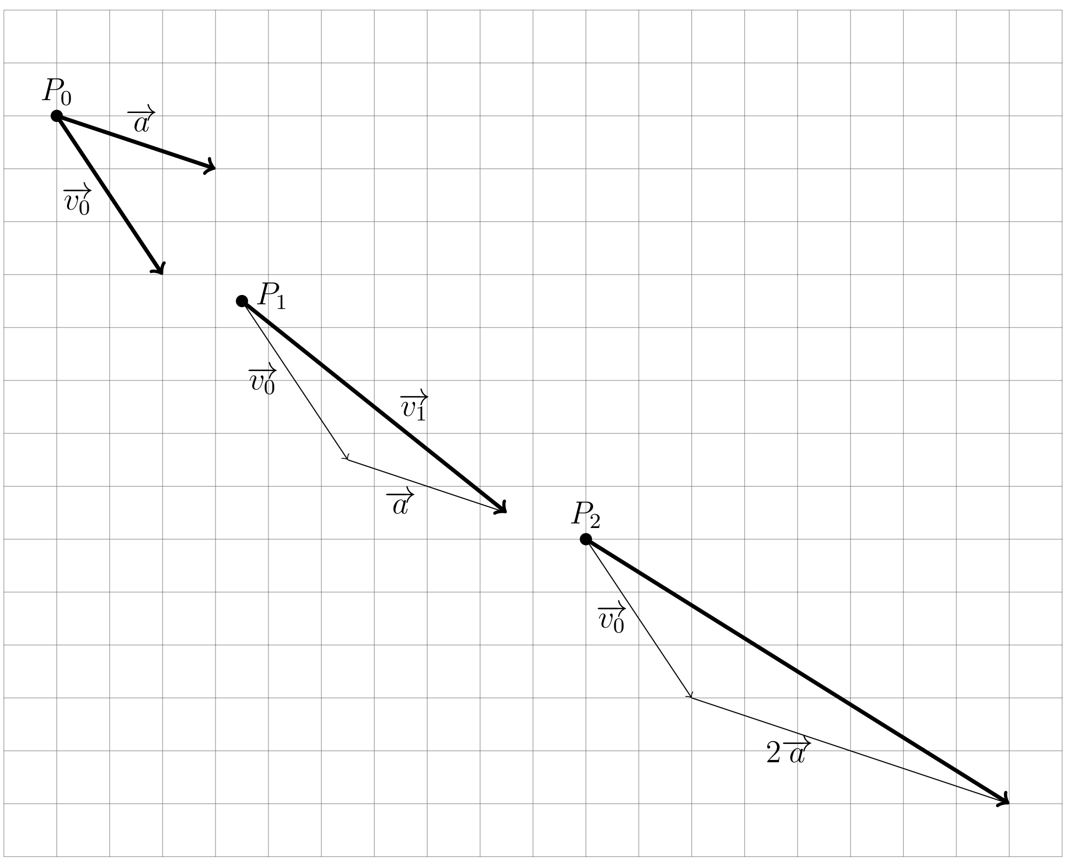
Problem 7 *Without using the Pythagoras' theorem, find the length of the diagonal of a square with the side length one.*



Acceleration

Acceleration is a vector that describes how velocity changes with time.

Consider a body, currently at P_0, having the velocity $\overrightarrow{v_0}$, in metres per second. Let $\overrightarrow{a}$ be the constant acceleration, in metres per second squared, the body undergoes. One second later, the body will be positioned at P_1 and will have the velocity vector $\overrightarrow{v_1} = \overrightarrow{v_0} + \overrightarrow{a}$. Two seconds later, the body will be positioned at P_2 and will have the velocity vector $\overrightarrow{v_2} = \overrightarrow{v_0} + 2\overrightarrow{a}$.



Some t seconds later, the body will have the velocity

$$\overrightarrow{v_t} = \overrightarrow{v_0} + t\,\overrightarrow{a}\,. \tag{1}$$

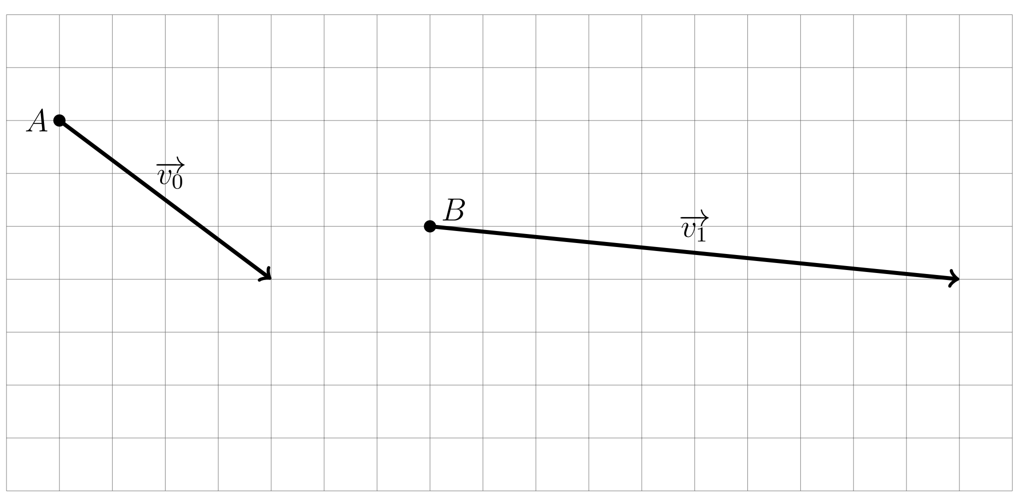
Using vector algebra, it is not hard to compute the position P_t either.

$$P_t = P_0 + t\overrightarrow{v_0} + \frac{t^2}{2}\overrightarrow{a} \tag{2}$$

It takes a bit of calculus to derive formula 2. We are not going to do it here.

Note that the word *acceleration* has two different meanings. One is a vector, as in the example above. The other meaning is the length of the vector that represents the magnitude of the velocity change. In this case, we do not put an arrow above the letter a representing acceleration. In other words, $a = |\overrightarrow{a}|$. Similarly, speed is the length of the velocity vector, $v = |\overrightarrow{v}|$.

Problem 8 *Moving with a constant acceleration, a body changes its position from point A to point B in one second. The velocities of the motion, in metres per second, are represented by the vectors $\overrightarrow{v_0}$ and $\overrightarrow{v_1}$. Find the acceleration vector.*

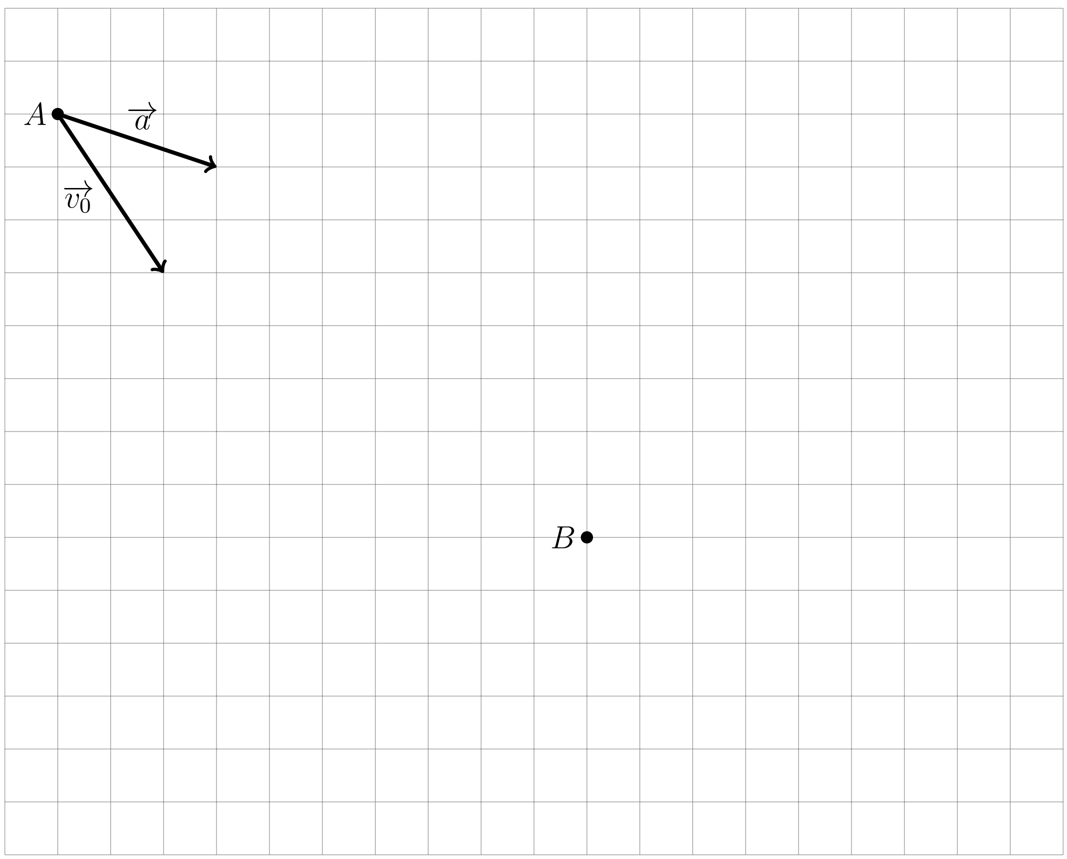


The side length of the grid squares above is one metre. Find the magnitude of the acceleration.

$$a =$$

Check if formula 2 correctly describes the motion shown on the above picture.

Problem 9 *At the initial moment of time, the body at point* A *is moving with the velocity vector* $\overrightarrow{v_0}$*, measured in mph, miles per hour. The constant acceleration acting on the body is represented by the vector* $\overrightarrow{a}$*, measured in miles per hour squared. Two hours later, the body is located at point* B*. Draw its velocity vector at the moment.*



The side length of the grid squares on the picture above is ten miles. Find the speed of the body at point B.

$$v_2 =$$

Check if formula 2 correctly describes the motion shown on the above picture.

The Second Law

$$\overrightarrow{F} = m\overrightarrow{a} \tag{3}$$

Here $\overrightarrow{F}$ is the net force acting on a body of a constant mass m and $\overrightarrow{a}$ is the acceleration the body undergoes as a result of the action. The mass measures the *inertia* of the body – the heavier the body the harder it is to change the way it moves.

In modern day physics, the force is measured in *newtons*. One newton is the force needed to accelerate 1 kilogram of mass at the rate of 1 metre per second squared.

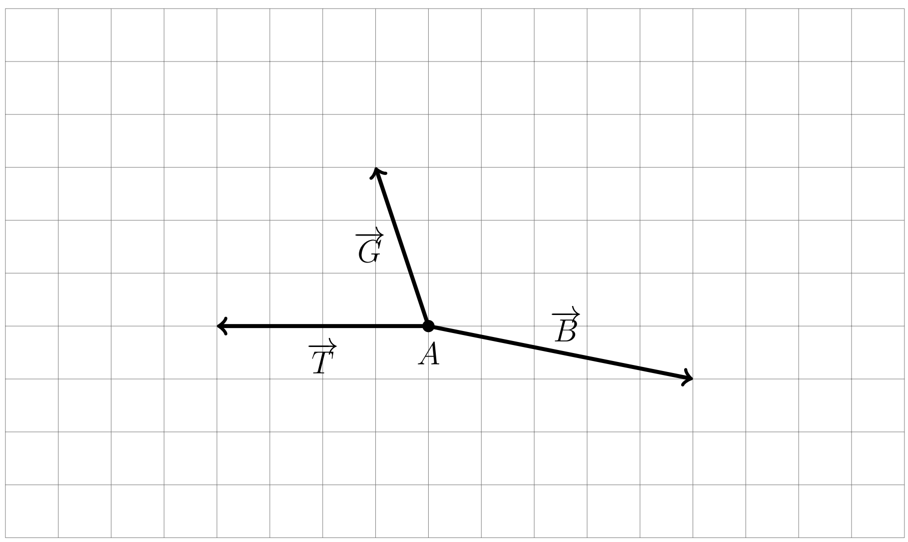
$$1N = 1kg\,\frac{m}{s^2} \tag{4}$$

In the US and Britain, they also use an outdated unit, one pound-force, 1 lbf. The pound-force is defined as the gravitational force exerted on a mass of one pound on the surface of the Earth. Our home planet is not a perfect sphere. What's more, its density varies for different locations on the globe. The gravitational pull of the Earth is not uniform. Hence, 1 lbf is only practical to use in calculations that do not require high precision.

The following formula helps to convert pounds of force to newtons.

$$1\ lbf \approx 4.448\ N \tag{5}$$

Problem 10 *The Millenium Falcon spaceship, at point A at the moment, is trying to escape from the Death Star that in its turn is trying to arrest the ship using the attracting beam. The thrust of the Falcon's engines, 200,000 kN in total, is represented by the vector* $\overrightarrow{T}$. *The force of the beam is represented by the vector* $\overrightarrow{B}$. *In addition, the neighbouring star exerts the gravitational pull* $\overrightarrow{G}$ *on the ship. Draw the vector of the net force acting on the vessel.*



The mass of the ship is 400 metric tons. What is the acceleration, in metres per second squared, the ship undergoes at the moment?

$a =$

The purpose of the daring mission was to find out the maximal force of the attracting beam. However, Captain Solo is not too good with vectors. Please help him complete the mission.

$$|\overrightarrow{B}| =$$

The Third Law

When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction to the force exerted on it by the first body.

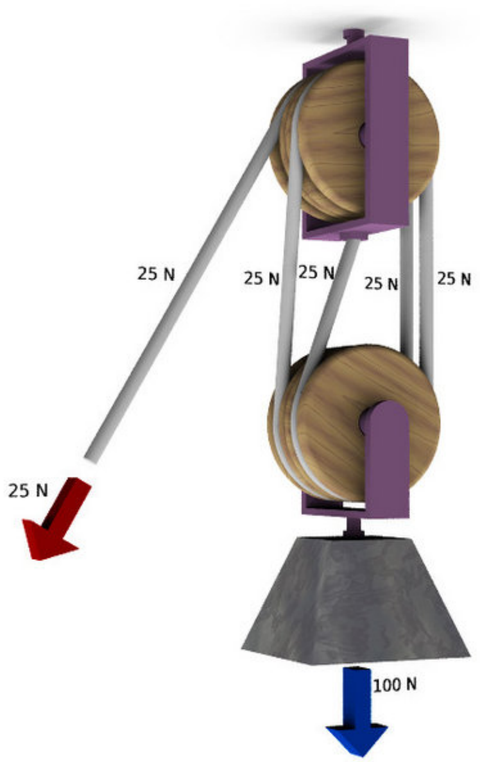
The vector form makes the Third Law very clear.

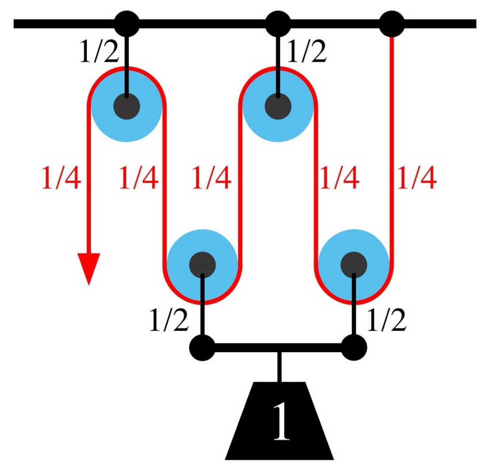
$$\overrightarrow{F}_{reaction} = -\overrightarrow{F}_{action} \tag{6}$$

Problem 11 *In a second, the (unmanned) truck and car on the picture below will collide in a crash test. The weight of the truck is 20 metric tons. The weight of the car is 2 metric tons. Find the ratio of the accelerations,* a_t *(acceleration of the truck) and* a_c *(that of the car), the vehicles will undergo.*

$$a_t \div a_c =$$

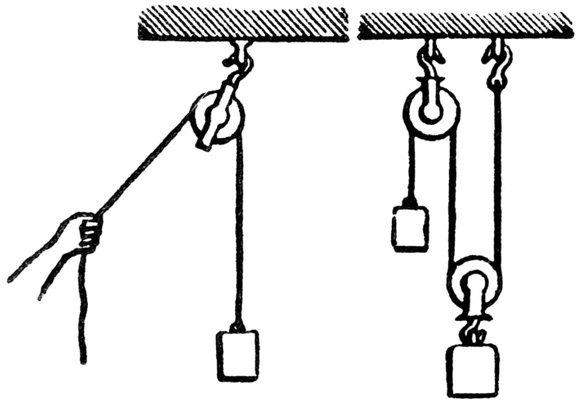
The pictures below show the workings of one of the simplest, and most important, machines invented by the humanity, a pulley. The first is a picture and schematics of a *double tackle*, the device with the *advantage* four.





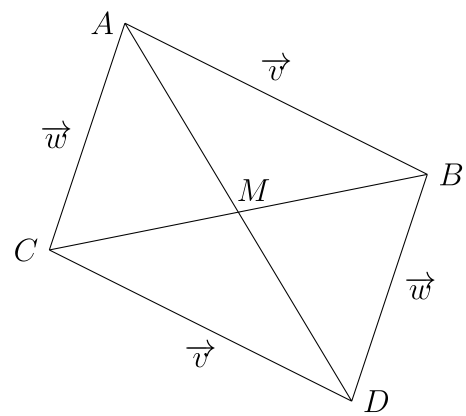
Problem 12 *Use the Third Law of Newton to explain how the double tackle works.*

Problem 13 *What is the advantage of the pulley on the left-hand side of the below picture? On the right-hand side?*

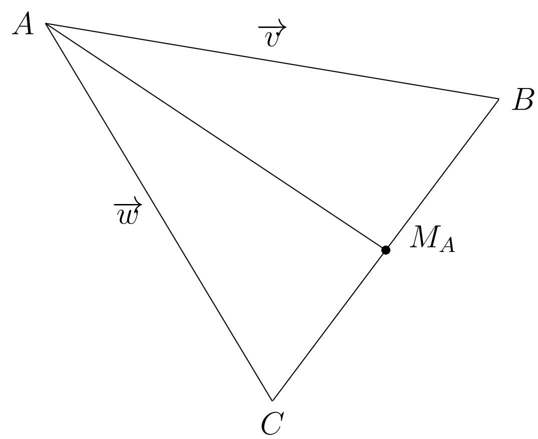


Problem 14 *Draw the scheme of a pulley with the advantage three.*

Problem 15 *Use vector algebra to prove that diagonals of a parallelogram in the Euclidean plane split each other in halves.*



Problem 16 *Use vector algebra to prove that all the three medians of a triangle in the Euclidean plane intersect at one point that splits each of the medians in the ratio 2:1 counting from the vertices.*



If you are finished doing all the above and there remains some time...

Problem 17 *Find the area of an equilateral triangle with the side length a.*

Problem 18 *Does there exist an equilateral triangle in the Euclidean plane such that all of its vertices have integral coordinates? Why or why not?*