HILBERT'S HOTELS (AN INTRODUCTION TO SET THEORY)

LAMC INTERMEDIATE/ADVANCED GROUPS - 2/09/14

(1) Write down the definitions of the following, and give an example when possible.

  (a) Two sets **Have the same number of elements:**

  Solution: If the elements from each set can be paired up.

  (b) A function between two sets is **One-to-One:**

  Solution: A function f is one-to-one if whenever $a \neq b$ we have $f(a) \neq f(b)$.

  (c) A function between two sets is **Onto:**

  Solution: A function $f : A \to B$ is onto if for every b in the set B we have some a in the set A so that $f(a) = b$. (Said more simply, everything in the target space is hit by something)

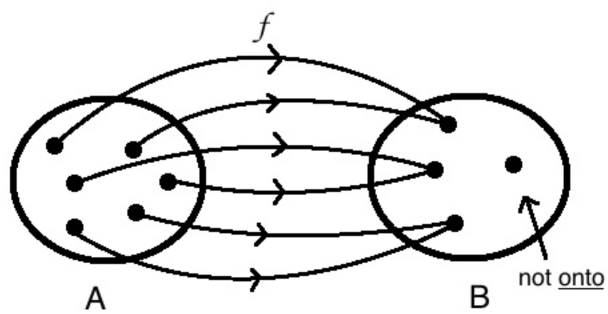
  (d) A function between two sets is a **Bijection:**

  Solution: A function is a bijection if it is both one-to-one and onto.

  (e) Give a more "grown-up" definition of two sets "having the same number of elements."

  Solution: If there is a bijection between the two sets.

(2) Here is a picture of a function that is not one-to-one or onto. (**Solutions will Vary**)



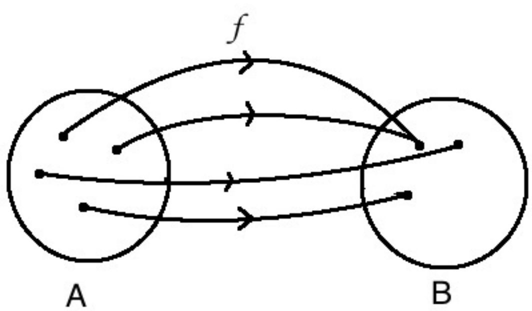
(a) Draw your own picture of a function that is onto but not one-to-one.

(b) Draw your own picture of a function that is one-to-one but not onto.

(c) Draw your own picture of a function that is one-to-one and onto.

(3) Determine which of the following functions are onto, one-to-one, and bijections.

(a) The function f represented pictorially as:



Solution: This function is onto but not one-to-one.

(b) The function $f(n) = n$ from $\mathbb{N}$ to $\mathbb{N}$.

(i) **Solution:** This function is one-to-one. Suppose $n \neq m$ then $f(n) = n \neq m = f(m)$ so it is one-to-one.

(ii) **Solution:** This function is onto. For any m in the target space, $m = f(m)$.

(c) The function $f(n) = n + 5$ from $\mathbb{N}$ to $\mathbb{N}$.

(i) **Solution:** This function is one-to-one. To see this suppose $n \neq m$, then

$$f(n) = n + 5 \neq m + 5 = f(m).$$

(ii) **Solution:** It is not onto. To see this, notice that there is no n such that $n + 5 = 0$.

(d) The function $f(n) = n \pmod 2$ from $\mathbb{N}$ to $\mathbb{N}$. What is the image of this function?

(i) **Solution:** This function is not one-to-one. For instance, $f(5) = f(7)$.

(ii) **Solution:** The function is also not onto. The image is simply $\{0, 1\}$ as all evens are sent to 0 and all odds are sent to 1.

(4) One-to-One functions, Onto functions, and Bijections.

(a) Can you come up with an onto function from the set $\{a, b, c, \cdots, z\}$ to the set $\{1, 2, 3, \cdots, 25\}$?

Solution: Yes. $a \mapsto 1, b \mapsto 2, \cdots, y \mapsto 25, z \mapsto 25$.

(b) What about an onto function from $\{1, 2, 3, \cdots, 25\}$ to $\{a, b, c, \cdots, z\}$?

Solution: No. There are fewer things in the first set, so it is impossible to be onto!

(c) Is there a bijection from $\{1, 2, 3, 4, \cdots, 10\}$ to $\{2, 4, 6, 8, \cdots, 20\}$? If so, give an explicit formula for the function.

Solution: Yes there is. The formula is $f(x) = 2x$.

(d) Is there a bijection from $\{2, 4, 6, 8, \cdots, 20\}$ to $\{1, 2, 3, 4, \cdots, 10\}$?

Solution: Yes, of course, it's just in the inverse of the previous one. $f(x) = \frac{x}{2}$.

(e) Is there a one-to-one function from the set of students in the room to the set of chairs in the room?

Solution: Yes, assuming everyone is sitting in different chairs, we can map each student to the chair they are sitting in.

(5) In this problem A is a set with n elements and B is a set with m elements (m and n are natural numbers).

(a) Suppose there is a one-to-one function $f : A \to B$. What (if anything) can we say about the relationship between m and n?

Solution: $n \leq m$ because we can now pair each element of A with an element of B, but we don't have to use all the elements of B.

(b) Suppose there is an onto function $f : A \to B$. What (if anything) can we say about the relationship between m and n?

Solution: We can now say that $n \geq m$. This follows because every element of B is hit by an element of A, so there must have been at least as many elements in A as there are in B.

(c) Suppose there is a bijection $f : A \to B$. What (if anything) can we say about the relationship between m and n?

Solution: Now we can say that $m = n$ by combining the previous two parts!

(d) Suppose $n \leq m$. Show that we can create an onto function $g : B \to A$.

Solution: List the elements of A as $a_1, \cdots, a_n$ and the elements of B as $b_1, \cdots, b_m$. Send $b_i \mapsto a_i$ if $i \leq n$ and $b_i \mapsto a_1$ if $i > n$. This is definitely onto!

(e) Suppose $n \geq m$. Show that we can create a one-to-one function $g : B \to A$.

Solution: The function described exactly as in (d) now becomes one-to-one (though its no longer onto).

(f) Suppose $n = m$. Show that we can create a bijection $g : A \to B$.

Solution: Once again, the function described in (d) now becomes a bijection.

(g) (Baby Cantor-Schroeder-Bernstein) Show that if we have a one-to-one function $f : A \to B$ and a one-to-one function $g : B \to A$ then we have a bijection $h : A \to B$. (**Draw a picture!)**

Solution: By parts (a) and (b) we know that $m = n$. But then by part (f) we know we can create a bijection $g : A \to B$.

(6) Using the definitions above, try to come up with your own definition of the word "**infinite.**"

Solution: My definition would be: A set A is infinite if there is an onto function $f : A \to \{0, 1, \cdots, n\}$ for every n. Alternatively, you could define finite and say infinite means not finite.

(7) Using your definition, prove rigorously that:

(a) $\mathbb{N}$ is infinite.

Solution: Using my definition this is very immediate. Fix a set $A = \{0, 1, \cdots, n\}$ and define $f : \mathbb{N} \to A$ by $f(x) = x$ if $x \leq n$ and $f(x) = 0$ if $x > n$.

(b) The set of Math Circle instructors is not infinite.

Solution: There is no onto function from the Math Circle instructors to the set $\{0, 1, \cdots, 5000000000000\}$ for instance.

(c) The set $\mathbb{Z}$ is infinite.

Solution: Use the same function as in (a) but send the negative integers to zero as well. To be more precise, $f : \mathbb{N} \to A$ by $f(x) = x$ if $0 \leq x \leq n$ and $f(x) = 0$ otherwise.

Hilbert's Hotel

Hilbert's Hotel is the most popular hotel in the galaxy, located just light-years away from the edge of the Milky Way. One day in the not-so-distant future a traveller from Earth arrived at the hotel:

"Are there any rooms vacant?"

"At the moment all rooms are occupied by other guests." Replied the hotel clerk.

"Is there anything you can do?" replied the traveller, "I really need a room for tonight."

The clerk replied, "Okay, here is your room key. Room number one."

Explanation.

(1) An explanation of this is as follows: The hotel has infinitely many rooms $1, 2, 3, \cdots$. When the traveller arrives, the hotel clerk asks the guest in room 1 to move into room 2, the guest in room 2 to move into room 3, etc. Explain this process using a one-to-one function.

Solution: The one-to-one function in question is $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 4, \cdots$ from $\mathbb{N}$ to itself. (This is weird, how can you have a one-to-one function from a set into itself that isn't onto!)

(2) Explain how the Hilbert Hotel can accomodate the following:

(a) All of the Math Circle instructors need rooms, but the hotel is full.

Solution: Have all the guests move up 5 rooms.

(b) New people, called $\{p_1, p_2, p_3, \cdots\}$, need rooms, but the hotel is full.

Solution: If a guest is in room n have him move to room $2n$ then guest p_i can take room $2i + 1$.

(3) Can you find bijections between the following sets? (**Hint:** Once again, it will be useful to draw diagrams)

(a) $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$ and $2\mathbb{N} = \{0, 2, 4, 6, \cdots\}$ (even numbers).

Solution: $f(n) = 2n$

(b) $\mathbb{N}$ and $2\mathbb{N} + 1 = \{1, 3, 5, \cdots\}$ (odd numbers).

Solution: $f(n) = 2n + 1$

(c) $\mathbb{N}$ and $\{A, 0, 1, 2, 3, \cdots\}$

Solution: $f(0) = A$ and $f(n) = n - 1$ if $n > 0$.

(d) $\mathbb{N}$ and $\mathbb{N}+5=\{5,6,7,\cdots\}$

Solution: $f(n)=n+5$.

(e) $\mathbb{N}$ and $\mathbb{Z}=\{0,1,-1,2,-2,3,-3,\cdots\}$ (**Hint:** It is easy to split up $\mathbb{Z}$ into two groups, positive and negative. Can you find a similar grouping for $\mathbb{N}$? Once you find the grouping, come up with a bijection.)

Solution: $f(n)=\frac{n}{2}$ if n is even. And $f(n)=-\frac{n+1}{2}$ if n is odd. (Solution by Vanessa!)

HOMEWORK

(1) Show that the set of evens and the set of odds have the same size.

(2) There is a sporting event near the Hilbert Hotel and all the teams need a place to stay. The hotel was originally empty, but there are infinitely many teams, and each team has infinitely many players. Can the Hilbert Hotel accomodate all the teams?

(**Hint:** You'll have to break up the hotel into infinitely many groups of infinite size. Note that there are infinitely many primes, and for example, the sets $\{2, 2^2, 2^3, \cdots\}$ and $\{3, 3^2, 3^3, \cdots\}$ are disjoint.)