# GEOMETRY & INEQUALITIES

LAMC INTERMEDIATE GROUP - 2/09/14

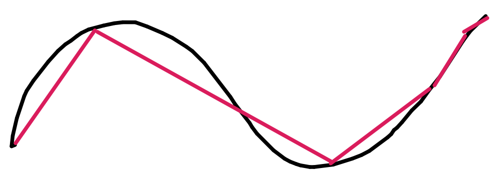
## The Triangle Inequality!

(1) State the Triangle Inequality. In addition, give an argument explaining why it should be true.

- Given the three side lengths of a triangle $a, b, c$ we have $a + b \geq c$.

(2) Now we will prove that the shortest path between two points is a straight line.

(a) Given any path between two points $a$ and $b$ in the plane, show that the path can be approximated by polygonal paths. (That is, explain how to approximate a path by polygonal paths. Use a picture)



(b) Now, using a limiting argument (remember sequences!) we can assume that any path is polygonal. Assuming this, show that the straight line is the shortest path connecting points $a$ and $b$.

The limiting argument is that the length of the curve is the limit of the lengths of polygonal lines. The polygonal lines make triangles, which are minimized by a straight line because of the triangle inequality.

**Math Kangaroo.**

(1) 28. We have four copies of this triangle covering the square.

(2) 24. The square has perimeter 16, and so does the triangle by assumption. Therefore in total the square plus the triangle have area 32. But the side which is not used in the total perimeter is counted twice and has length 4. So we must subtract 8.

**Inequalities amongst triangles!**

(1) Out of all triangles with two sides $a$ and $b$ fixed, find the triangle with maximum area.

- Hint: The area of a triangle is $\frac{1}{2}$(base)(height).
  - If we fix $a$ to be the base and $b$ connected to $a$ the height will be largest if $b$ is at a right angle to $a$.

(2) Among all triangles inscribed in a fixed circle, with a fixed base $b$, find the triangle with maximum area.

- Hint: Draw a picture!
  - The maximum area is simply the one with the largest height. See the picture on problem three (but change the ellipse to a circle).

(3) Out of all triangles with a fixed perimeter $P$ and base $b$, find the the triangle with maximum area.

- Hint: The following picture makes the argument very similar to problem 2.
  - The height needs to be maximized again, giving us an isosceles triangles.

(4) Show that your solution to problem number 3 must also be a solution to the following problem. Amongst all triangles with area $\mathcal{A}$ and base $b$, which is the triangle with the smallest perimeter?

- Hint: Don't solve the problem again. Use the fact that this triangle is a solution to problem 3.
  - Suppose there is a non-isosceles solution. Then we can take an isosceles triangle of the same perimeter and base $b$, this triangle now has larger area $A'$. But because of this we can then find an isosceles triangle with smaller perimeter and area $A$ a contradiction.

(5) Out of all triangles with perimeter $P$, which has the largest area?

An equilateral triangle. Let triangle $ABC$ be the solution. Then the area is maximized as a triangle with fixed base $AB$ and hence the two sides $BC$ and $AC$ have the same length. Similarly, we see the other side is also the same.

**Inequalities amongst rectangles!**

(1) To start we want to prove an algebraic inequality. Prove that for $a, b \geq 0$ we have

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

(This is a special case of the AM-GM inequality.)

- Hint: Show that it suffices to show the inequality for $a$ and $b$ replaced by $a^2$ and $b^2$. Then use the fact that for any $z$ we know that $z^2 \geq 0$

  Consider $q^2 = a$ and $p^2 = b$ we can see it suffices to consider the replacement. Now with the replacement we notice that: $(p-q)^2 \geq 0$ and $(p-q)^2 = p^2 + q^2 - 2pq$ and so $pq \leq \frac{p^2+q^2}{2}$ as desired.

(2) Among all rectangles with perimeter $P$, find the rectangle with the largest area.

The area is $A = lw$, length times width. So by the above inequality we know that

$$\sqrt{A} \leq \frac{l+w}{2} = P/4$$

where $P$ is the perimeter. Equality occurs only when $l = w$ so we see that we have the maximum when the rectangle is a square.

(3) Among all rectangles with area $A$, find the rectangle with the smallest perimeter.

See problem $2$, or problem $3$ from the triangle section.

**An inequality in an $n$-gon.**

(1) Among all quadrilaterals with perimeter $P$, show that the quadrilateral with maximum area is equilateral.

- Hint: You will want to use an inequality you know about triangles.
  - Take one of the diagonals of the quadrilateral. The area of the two triangles created are maximized for the fixed base (the diagonal) and hence they are each equilateral. Performing this for both diagonals yields that the quadrilateral is equilateral.

(2) Among all $n$-gons with perimeter $P$, show that the $n$-gon with maximum area is equilateral.

- Hint: Try to replicate your argument from the previous problem.
  - The argument is exactly the same as above, but make many more triangles.

**The Shortest Distance... with a twist.** We learned at the beginning of the day that the shortest path between two points on the plane is a straight line. Now we want to examine shortest paths with a few constraints.

(1) A town contains two circular ponds. The first pond has radius $2$ and center at $(0, 2)$ (so it is tangent to the $x$ - axis). The second pond has radius $5$ and center at $(15, 10)$. What is the shortest distance between the edges of the ponds. (Prove your answer)

- Hint: Use the triangle inequality!
  - The shortest distance comes from drawing a line directly between the two centers and taking the resulting path between edges. This follows from the triangle inequality as in this picture: So the length is $10$ since $8, 15, 17$ is a pythagorean triple, giving us the length as $17 - 5 - 2 = 10$.

(2) A town contains a circular pond with radius $2$ and center $(0, 4)$. It also has a horizontal river located on the $x$-axis. Derek's house is located at $(7, 20)$. Because of the recent drought, Derek needs to walk to the river and fill up a bucket, then walk to the pond and empty the bucket. Describe the shortest path Derek can take geometrically (i.e. with a picture). What is the length of this path?

Apply the above problem, reflect the circle across the river and take the shortest path as above. Then reflect the path back. The length is $25 - 2 = 23$ as $7, 24, 25$ is a pythagorean triple.

(3) **(Hard)** Dustin and Morgan live on opposite corners of a $4$-way intersection. The roads must be crossed perpendicular to the street. The horizontal road is located between the lines $y = 1$ and $y = -1$ and the vertical road is located between the lines $x = -1$ and $x = 1$. Dustin's house is located at $(-3, -3)$ and Morgan's house is located at $(4, 11)$. Describe the shortest path between Dustin's house and Morgan's house geometrically. What is the length of the path?

This is similar to the single road problem we did in class. Smash the two roads together and Morgan's house becomes $(2, 9)$, now draw the straight line connecting them and fatten out the roads again. This line has length $13$ because $5, 12, 13$ is a Pythagorean triple. So the length is $13 + 2 + 2 = 17$ ($2$ for each of the roads, and $13$ for the length of the straight line path).

(4) **(Very Hard)** Given two points $A$ and $B$ lying on the same side of a river (on the $x$-axis). Can you find a point $P$ on the river such that the angle between the line $BP$ and the river is exactly half the angle between the line $AP$ and the river?

This is by far the hardest problem on the sheet. See this picture for explanation: