

GEOMETRY & INEQUALITIES

LAMC INTERMEDIATE GROUP - 2/23/14

The Triangle Inequality!

(1) State the Triangle Inequality. In addition, give an argument explaining why it should be true.

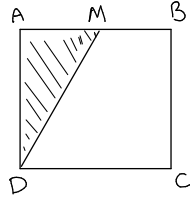
(2) Now we will prove that the shortest path between two points is a straight line.

(a) Given any path between two points a and b in the plane, show that the path can be approximated by polygonal paths. (That is, explain how to approximate a path by polygonal paths. Use a picture)

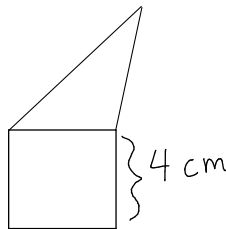
(b) Now, using a limiting argument (remember sequences!) we can assume that any path is polygonal. Assuming this, show that the straight line is the shortest path connecting points a and b .

A couple of Math Kangaroo problems to refresh our memory about area and perimeter.

- (1) (Math Kangaroo) In the figure below, $ABCD$ is a square. The area of the shaded triangle (ADM) is 7 cm, and M is the midpoint of AB . What is the area of the whole square?



- (2) (Math Kangaroo) In the figure below, the square and triangle have the same perimeter. What is the perimeter of the whole figure?



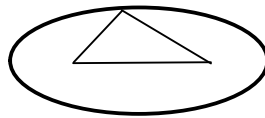
A few more problems to refresh your memory about perimeter and area.

- (1) We all know the area of a rectangle is $\mathcal{A} = (\text{length})(\text{width})$. Using the fact that two identical right triangles can be put together to form a rectangle, derive a formula for the area of a right triangle.

- (2) Prove that the formula you found in problem 1 also holds for non-right triangles.

- (3) Can you find two rectangles with the same perimeter and different areas? What about two triangles?

- (4) Suppose we have two fixed points p and q and a string tied to them at each end. If the string is pulled taut, so as to form a triangle, the shape the vertex of this triangle traces out is called an ellipse:



Explain why given any base b and fixed perimeter P we can find an ellipse that contains every (non-degenerate) triangle with perimeter P and base b .

Inequalities amongst rectangles!

- (1) To start we want to prove an algebraic inequality. Prove that for $a, b \geq 0$ we have

$$\sqrt{ab} \leq \frac{a+b}{2}.$$

- Hint: First multiply the inequality by 2 and square both sides to show that it is equivalent to another inequality. Then use the fact that for any z we know that $z^2 \geq 0$

- (2) Among all rectangles with perimeter P , find the rectangle with the largest area.

- (3) Among all rectangles with area \mathcal{A} , find the rectangle with the smallest perimeter.

Inequalities amongst triangles!

(1) Out of all triangles with two sides a and b fixed, find the triangle with maximum area.

(2) Among all triangles inscribed in a fixed circle, with a fixed base b , find the triangle with maximum area.

- Hint: Draw a picture!

(3) Out of all triangles with a fixed perimeter P and base b , find the triangle with maximum area. Can you find a (non-degenerate) triangle with minimum area?

- Hint: Use the ellipse you constructed earlier!

(4) Show that your solution to problem number 3 must also be a solution to the following problem. Amongst all triangles with area \mathcal{A} and base b , which is the triangle with the smallest perimeter?

- Hint: Don't solve the problem again. Use the fact that this triangle is a solution to problem 3.

(5) Out of all triangles with perimeter P , which has the largest area? Is there a (non-degenerate) triangle with minimum area?

An inequality in an n -gon.

(1) Among all quadrilaterals with perimeter P , show that the quadrilateral with maximum area is equilateral.

- Hint: You will want to use an inequality you know about triangles.

(2) Among all n -gons with perimeter P , show that the n -gon with maximum area is equilateral.

- Hint: Try to replicate your argument from the previous problem.

The Shortest Distance... with a twist. We learned at the beginning of the day that the shortest path between two points on the plane is a straight line. Now we want to examine shortest paths with a few constraints.

- (1) A town contains two circular ponds. The first pond has radius 2 and center at $(0, 2)$ (so it is tangent to the x - axis). The second pond has radius 5 and center at $(15, 10)$. What is the shortest distance between the edges of the ponds. (Prove your answer)

- Hint: Use the triangle inequality!

- (2) A town contains a circular pond with radius 2 and center $(0, 4)$. It also has a horizontal river located on the x -axis. Derek's house is located at $(7, 20)$. Because of the recent drought, Derek needs to walk to the river and fill up a bucket, then walk to the pond and empty the bucket. Describe the shortest path Derek can take geometrically (i.e. with a picture). What is the length of this path?

- (3) **(Hard)** Dustin and Morgan live on opposite corners of a 4-way intersection. The roads must be crossed perpendicular to the street. The horizontal road is the strip between the lines $y = 1$ and $y = -1$ and the vertical road is the strip between the lines $x = -1$ and $x = 1$. Dustin's house is located at the point $(-3, -3)$. Morgan's house is much bigger and can't be fit in a single point. His house is the square with verices located at $(4, 11)$, $(4, 12)$, $(5, 12)$, and $(5, 11)$. Describe the shortest path between Dustin's house and Morgan's house geometrically. What is the length of the path?

- (4) **(Very Hard)** Given two points A and B lying on the same side of a river (on the x -axis). Can you find a point P on the river such that the angle between the line BP and the river is exactly half the angle between the line AP and the river?

HOMEWORK

(1) Triangle T_1 has sidelengths 5, 6, 7 and triangle T_2 has sidelengths 6, 6, 6. Which triangle has larger area?

(2) Quadrilateral A has sidelengths 4, 5, 7, 8 and quadrilateral B has sidelengths 6, 6, 6, 6. Which quadrilateral has larger area?