SEQUENCES

MATH CIRCLE (HS1) 2/23/2014

A *sequence* is an ordered list of elements. These elements are often numbers (but not always numbers).

A sequence can either be finite or infinite. If a sequence is finite, with *length* l, we'll denote it:

$$\langle a_0, a_1, \ldots, a_{l-1} \rangle$$

where a_n is the nth element of the sequence. Otherwise, a sequence is infinite, and denoted:

$$\langle a_0, a_1, a_2, \ldots \rangle$$

for all $n \in \mathbb{N}$.

For example, the sequence of even numbers is $\langle 0, 2, 4, 6, 8, \ldots \rangle$.

We can give an *explicit definition* of this sequence as $\langle a_n | n \in \mathbb{N} \rangle$ where $a_n = 2 \cdot n$.

We can also give a *recursive definition* of this sequence as $\langle a_i | i \in \mathbb{N} \rangle$ where $a_0 = 0$ and $a_{n+1} = a_n + 2$.

0) Write out the first 8 terms of the following sequences:

a) The prime numbers (in increasing order).

b) The Fibonacci numbers, with recursive definition $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$.

c) The sequence with explicit definition $a_n = n(n+1)/2$.

d) The sequence of squares (in increasing order).

e) The sequence of number of diagonals in a regular n-gon.

f) Number of ways to color the vertices of a regular n-gon with 2 colors

g) The sequence with recursive definition $a_0 = 1, a_1 = 2$ and $a_{n+2} = a_{n+1} + a_n$.

h) Number of ways to color the vertices of a regular n-gon with 2 colors so that adjacent vertices are different colors.

i) The sequence with recursive definition $a_0 = 1, a_1 = 3$ and $a_{n+2} = 2(a_{n+1} + a_n)$.

j) The sequence with explicit definition $a_n = n(n+1)(2n+1)/6$.

k) Number of ways to color the vertices of a triangle with n colors so that adjacent vertices are different colors.

l)* Number of ways to color the vertices of a square with n colors so that adjacent vertices are different colors.

Two of the most important types of sequences are *arithmetic* and *geometric* sequences. Arithmetic sequences have the property that $a_{i+1} - a_i$ is constant, while geometric sequences have the property that a_{i+1}/a_i is constant.

1) Come up with both explicit and recursive definitions for the following arithmetic or geometric sequences:

a) $\langle 2, 6, 10, 14, 18, \ldots \rangle$

b) $\langle 26, 17, 8, \ldots \rangle$

c) $\langle c, c + k, c + 2k, \ldots \rangle$

d) $\langle 5, 15, 45, \ldots \rangle$

e) $\langle 28, 14, 7, \ldots \rangle$

f) $\langle c, ck, ck^2, \ldots \rangle$

The Method of Mathematical Induction (MMI) is a way to show that a property $P(n)$ holds for all $n \in \mathbb{N}$. A proof using MMI consists of two steps:

Basis: Show that $P(0)$ holds.

Induction: Assume $P(k)$ holds, and show that $P(k + 1)$ holds.

Then, MMI tells us that $P(n)$ holds for all $n \in \mathbb{N}$.

MMI is a very useful tool for dealing with sequences.

2) Find (with proof!) explicit definitions for the following sequences:

a) The sequence with recursive definition $a_0 = c$ and $a_{n+1} = a_n + k$.

b) The sequence with recursive definition $a_0 = c$ and $a_{n+1} = a_n \cdot k$.

c) The sequence with recursive definition $a_0 = 0$ and $a_{n+1} = a_n + n$.

Hint: You've seen this sequence already!

d) The sequence with recursive definition $a_0 = 0$ and $a_{n+1} = a_n + n^2$.

3) Find (with proof!) a recursive and explicit definition for the sequence of number of diagonals in a regular n-gon.

Hint: Find and prove a recursive definition, and use it to prove the explicit definition.

4) Find (with proof!) a recursive and explicit definition for the sequence of the number of binary words avoiding "00".

5) Find (with proof!) a recursive definition of the sequence where $a_n =$ the number of sequences of 1's and 2's which add up to n.

For example, $\langle 1, 1 \rangle, \langle 2 \rangle$ are the only two sequences which add up to 2, while $\langle 1, 1, 1 \rangle, \langle 2, 1 \rangle, \langle 1, 2 \rangle$ are the three sequences which add up to 3.

6)* Find (with proof!) a recursive definition for the sequence where $a_n =$the number of Dyck words of length $2n$. A *Dyck word* is a binary word with n 0's and n 1's so that no initial portion of the word has more 1's than 0's.

For example, 01 is the only Dyck word of length 2, and $0011, 0101$ are the only Dyck words of length 4.