

# Bonus problems

## Math Circle (Advanced)

February 24, 2014

### 1. Equilateral triangles on graph paper.

- (a) On the piece of graph paper below, try to draw an equilateral triangle whose vertices all lie at intersecting points. (If you think you can't do it, make sure you've tried several different angles, and the like!)



Figure 1: graph paper

- (b) Prove that for any triangle with vertices on the lattice (intersection) points of a piece of graph paper with square grid, its doubled area is an integer.
- (c) Find the length of the altitude of an equilateral triangle with side length  $a$  and find the triangle's area.
- (d) Prove that the square of the length of any segment with its endpoints on two lattice points is an integer.
- (e) Prove that if there exists an equilateral triangle with its vertices on three lattice points, then  $\sqrt{3}$  is a rational number.  
(Hint: what does it mean to be a rational number?)
- (f) Prove that if the square of a fraction, whose numerator and denominator are relatively prime, is an integer, then the denominator is 1.

- (g) Prove that what you were trying to do is impossible: there is no equilateral triangle with its vertices on the lattice points of a piece of graph paper with square tiles.
  - (h) Can you find a kind of graph paper with rectangular tiles that would make your assignment possible?
2. Numbers are placed on the vertices of a 100-gon in such a way that each of them equals the arithmetic mean of its neighbors. Prove that the values of all the numbers must be equal.
  3. Prove the same fact as in the previous problem if the numbers are placed on the squares of a chessboard and it is known that each of them does not exceed the arithmetic mean of its neighbors.