LINEAR TRANSFORMATIONS

MATH CIRCLE (HS1) 2/9/2014

Review

Recall how to multiply matrices and a matrix by a vector:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}.$$

The determinant of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is $\det(A) = ad - bc$. We also have that

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where A^{-1} is the inverse of A, i.e. $A \cdot A^{-1} = A^{-1} \cdot A = I_2$.

We have the following facts:

- A is *invertible* (i.e. A has an inverse) if and only if $\det(A) \neq 0$.
- $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$. In particular, $(A^{-1})^{-1} = A$ and $(A^T)^{-1} = (A^{-1})^T$.
- $\det(A^T) = \det(A)$ and $\det(kA) = k^2 \det(A)$.
- $\det(A \cdot B) = \det(A) \cdot \det(B)$. In particular, $\det(A^{-1}) = 1/\det(A)$.

We say that $\vec{z}$ is a *linear combination* of $\vec{x}, \vec{y}$ if $\vec{z} = c\vec{x} + d\vec{y}$ for real numbers c, d. A linear combination is *non-trivial* if c, d are not both zero.

We call $\vec{x}, \vec{y}$ *linearly dependent* if $\vec{0}$ is a non-trivial linear combination of $\vec{x}, \vec{y}$. Recall we showed last week that we can equivalently say $\vec{x}, \vec{y}$ are *linearly dependent* if and only if $\vec{x}$ is a multiple of $\vec{y}$ or $\vec{y}$ is a multiple of $\vec{x}$.

We say that $\mathcal{B} = \{\vec{x}, \vec{y}\}$ is a *basis* if any vector $\vec{z}$ can be written as a linear combination of $\vec{x}$ and $\vec{y}$.

Theorem 1: The following are equivalent, with $\mathcal{B} = \{\vec{x}, \vec{y}\}$ and $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{y} = \begin{bmatrix} c \\ d \end{bmatrix}$:

- $\mathcal{B}$ is a basis
- $\vec{x}$ and $\vec{y}$ are linearly independant
- $\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) \neq 0$

Examples

Both $\mathcal{B}_0 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ and $\mathcal{B}_1 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ are bases. Why? Using Theorem 1, we have that

$$\det\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0 \text{ and } \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 1 \neq 0$$

so they are bases. Or, using the original definition we have that (for any a, b real numbers):

$$\vec{z} = \begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \vec{z} = \begin{bmatrix} a \\ b \end{bmatrix} = (a-b)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We call $\mathcal{B}_0$ the *standard basis*.

Change of Bases

If $\mathcal{B} = \{\vec{x}, \vec{y}\}$ is a basis, we define a *vector with respect to* $\mathcal{B}$:

$$\vec{v_\mathcal{B}} = \begin{bmatrix} a \\ b \end{bmatrix}_\mathcal{B} = a\vec{x} + b\vec{y}.$$

Note, if $\mathcal{B}_0 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ (called the standard basis), then $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{B}_0}$.

Vectors with respect to a fixed basis behave exactly the same way as before!

Let $\mathcal{B}_0, \mathcal{B}_1$ be as above, and $\mathcal{B}_2 = \{\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\}, \mathcal{B}_3 = \{\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\}, \mathcal{B}_4 = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

As an example, we have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_0} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{B}_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}_4}$$

1) a) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_4}$ as a vector with respect to $\mathcal{B}_0$.

b) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_4}$ as a vector with respect to $\mathcal{B}_1$.

c) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_3}$ as a vector with respect to $\mathcal{B}_3$.

d) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_3}$ as a vector with respect to $\mathcal{B}_2$.

e) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_2} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_2}$ as a vector with respect to $\mathcal{B}_4$.

Suppose $\mathcal{B}, \mathcal{B}'$ are bases.

We say that C is a *change of basis matrix for $\mathcal{B}$ to $\mathcal{B}'$* if for all vectors $\vec{v}_{\mathcal{B}} = \vec{v_{\mathcal{B}'}}$, $C\vec{v_{\mathcal{B}}} = \vec{v_{\mathcal{B}'}}$.

Theorem 2: Suppose $\mathcal{B} = \{\vec{x}, \vec{y}\}, \mathcal{B}' = \{\vec{z}, \vec{w}\}$ are bases and let $C^{\mathcal{B}}_{\mathcal{B}'}$ denote the change of basis matrix for $\mathcal{B}$ to $\mathcal{B}'$. If $\vec{z} = a\vec{x} + b\vec{y}$ and $\vec{w} = c\vec{x} + d\vec{y}$ then

$$C^{\mathcal{B}}_{\mathcal{B}'} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} (\vec{z})_{\mathcal{B}} & (\vec{w})_{\mathcal{B}} \end{bmatrix}.$$

Fact: We also have that $C^{\mathcal{B}'}_{\mathcal{B}} = (C^{\mathcal{B}}_{\mathcal{B}'})^{-1}$ and if $\mathcal{B}''$ is a third basis, then $C^{\mathcal{B}}_{\mathcal{B}''} = C^{\mathcal{B}'}_{\mathcal{B}''} \cdot C^{\mathcal{B}}_{\mathcal{B}'}$.

2) a) Find $C^{\mathcal{B}_0}_{\mathcal{B}_1}, C^{\mathcal{B}_0}_{\mathcal{B}_2}, C^{\mathcal{B}_0}_{\mathcal{B}_3}, C^{\mathcal{B}_0}_{\mathcal{B}_4}$. Hint: This is easy once you understand Theorem 2!

b) Find $C^{\mathcal{B}_3}_{\mathcal{B}_1}$ using the formula described in Theorem 2. Then check your answer using a) and the above fact.

c) Find $C^{\mathcal{B}_2}_{\mathcal{B}_4}$, again using both ways.

Linear Transformations

Suppose T is a transformation of the plane. We say that T is a linear transformation if

$$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$$

for all real numbers a, b and vectors $\vec{x}, \vec{y}$.

Fact: If T is a linear transformation and $\mathcal{B} = \{\vec{x}, \vec{y}\}$ is a basis, then there is a matrix $M = M^{\mathcal{B}}$, called the *matrix of T with respect to $\mathcal{B}$*, so that

$$T(a\vec{x} + b\vec{y}) = \left(M \begin{bmatrix} a \\ b \end{bmatrix}\right)_{\mathcal{B}}.$$

In fact, if $T(\vec{x}) = a\vec{x} + c\vec{y}$, $T(\vec{y}) = b\vec{x} + d\vec{y}$ then

$$M^{\mathcal{B}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

3) Let $\mathcal{B} = \mathcal{B}_0$ throughout this problem. Find the matrix of T with respect to $\mathcal{B}$ for the following linear transformations. Hint: The fact above tells us that we just need to figure out how T acts on the basis!

a) Rotation by $90°$ counterclockwise.

b) Reflection about $y = 0$.

c) Reflection about $x = 0$.

d) Rotation by $180°$ counterclockwise.

e) Reflection about $y = x$.

f) Stretching by a factor of c.

g) Stretching by a factor of c, and then rotation by $90°$ counterwise. Do you see a relationship with your answer and the answers to a)+f)?

4) Find the matrix of T with respect to $\mathcal{B}$ for the following linear transformations. Note you get to choose which basis you use! Hint: Choose wisely!

a) The linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

b) The linear transformation such that

$$T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

c) The linear transformation such that

$$T\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, T\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

d) Stretching by a factor of 3 and reflecting about $y = -x$.

5) For each of the transformations in 4), find the matrix of T with respect to $\mathcal{B}_0$. Hint: Theorem 2 can be put to good use!