WHAT IS A BASIS?

MATH CIRCLE (HS1) 2/2/2014

Recall how to multiply matrices and a matrix by a vector:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix}.$$

The determinant of $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is $\det(A) = ad - bc$. We also have that

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where A^{-1} is the inverse of A, i.e. $A \cdot A^{-1} = A^{-1} \cdot A = I_2$.

We have the following facts:

- A is *invertible* (i.e. A has an inverse) if and only if $\det(A) \neq 0$.
- $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$. In particular, $(A^{-1})^{-1} = A$ and $(A^T)^{-1} = (A^{-1})^T$.
- $\det(A^T) = \det(A)$ and $\det(kA) = k^2 \det(A)$.
- $\det(A \cdot B) = \det(A) \cdot \det(B)$. In particular, $\det(A^{-1}) = 1/\det(A)$.

Linear Combinations

We say that $\vec{z}$ is a *linear combination* of $\vec{x}, \vec{y}$ if $\vec{z} = c\vec{x} + d\vec{y}$ for real numbers c, d.

A linear combination is *non-trivial* if c, d are not both zero.

For example, $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as

$$\begin{bmatrix} 4 \\ 5 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1) Write $\vec{y}$ as a linear combination of $\vec{x}_1, \vec{x}_2$:

a) $\vec{y} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

b) $\vec{y} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

c) $\vec{y} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

d) $\vec{y} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hint: Do $\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ first.

Linear Independence and Bases

We say vectors $\vec{x}, \vec{y}$ are *linearly dependent* if $\vec{0}$ is a non-trivial linear combination of $\vec{x}, \vec{y}$. Otherwise, they are called *linearly independent*.

2) a) Give an example of vectors that are linearly dependent.

b) Give an example of vectors that are linearly independent.

3) Prove that $\vec{x}$ and $\vec{y}$ are linearly dependent if and only if $\vec{x}$ is a multiple of $\vec{y}$ or $\vec{y}$ is a multiple of $\vec{x}$.

We say that $\mathcal{B} = \{\vec{x}, \vec{y}\}$ is a *basis* if any vector $\vec{z}$ can be written as a linear combination of $\vec{x}$ and $\vec{y}$.

Therefore, you have already shown that both $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ form bases (see Problem 1).

Theorem 1: The following are equivalent, with $\mathcal{B} = \{\vec{x}, \vec{y}\}$ and $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{y} = \begin{bmatrix} c \\ d \end{bmatrix}$:

- $\mathcal{B}$ is a basis
- $\vec{x}$ and $\vec{y}$ are linearly independant
- $\det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) \neq 0$

4) Are each of the following a basis?

If yes, prove your answer using the original definition of a basis.

Hint: Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination first.

a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

c) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

If $\mathcal{B} = \{\vec{x}, \vec{y}\}$ is a basis, we define a *vector with respect to* $\mathcal{B}$:

$$\vec{v_{\mathcal{B}}} = \begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{B}} = a\vec{x} + b\vec{y}.$$

Note, if $\mathcal{B}_0 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ (called the standard basis), then $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{B}_0}$.

Vectors with respect to a fixed basis behave exactly the same way as before!

Let $\mathcal{B}_1 = \{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}, \mathcal{B}_2 = \{\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}\}, \mathcal{B}_3 = \{\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}\}, \mathcal{B}_4 = \{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$.

As an example, we have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_0} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{B}_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}_4}$$

5) a) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_4}$ as a vector with respect to $\mathcal{B}_0$.

b) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_4}$ as a vector with respect to $\mathcal{B}_1$.

c) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_3}$ as a vector with respect to $\mathcal{B}_3$.

d) Write $\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}_3} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{\mathcal{B}_3}$ as a vector with respect to $\mathcal{B}_2$.

Suppose $\mathcal{B}, \mathcal{B}'$ are bases.

We say that C is a *change of basis matrix for $\mathcal{B}$ to $\mathcal{B}'$* if for all vectors $\vec{v}_{\mathcal{B}} = \vec{v}_{\mathcal{B}'}$, $C\vec{v}_{\mathcal{B}} = \vec{v}_{\mathcal{B}'}$.

Theorem 2: Suppose $\mathcal{B} = \{\vec{x}, \vec{y}\}, \mathcal{B}' = \{\vec{z}, \vec{w}\}$ are bases and let $C^{\mathcal{B}}_{\mathcal{B}'}$ denote the change of basis matrix for $\mathcal{B}$ to $\mathcal{B}'$. If $\vec{z} = a\vec{x} + b\vec{y}$ and $\vec{w} = c\vec{x} + d\vec{y}$ then

$$C^{\mathcal{B}}_{\mathcal{B}'} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} (\vec{z})_{\mathcal{B}} & (\vec{w})_{\mathcal{B}} \end{bmatrix}.$$

Fact: We also have that $C^{\mathcal{B}'}_{\mathcal{B}} = (C^{\mathcal{B}}_{\mathcal{B}'})^{-1}$ and if $\mathcal{B}''$ is a third basis, then $C^{\mathcal{B}}_{\mathcal{B}''} = C^{\mathcal{B}'}_{\mathcal{B}''} \cdot C^{\mathcal{B}}_{\mathcal{B}'}$.

6) a) Find $C^{\mathcal{B}_0}_{\mathcal{B}_1}, C^{\mathcal{B}_0}_{\mathcal{B}_2}, C^{\mathcal{B}_0}_{\mathcal{B}_3}, C^{\mathcal{B}_0}_{\mathcal{B}_4}$. Hint: This is easy once you understand Theorem 2!

b) Find $C^{\mathcal{B}_3}_{\mathcal{B}_1}$ using the formula described in Theorem 2. Then check your answer using a) and the above fact.

c) Find $C^{\mathcal{B}_2}_{\mathcal{B}_4}$, again using both ways.

Challenge Problems:

1) Prove the Fact following Theorem 2.

2) Prove Theorem 2.

3) Prove Theorem 1.