INTRODUCTION TO LINEAR ALGEBRA

MATH CIRCLE (HS1) 1/26/2014

Today we'll be mostly dealing with 2×2 and 2×1 matrices. Recall how we multiply matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \text{ and } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} ae+bf \\ ce+df \end{bmatrix},$$

as well as the transpose:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

(Take a second to convince yourself that these matches the definition from last week.)

We define the identity ($n \times n$) matrix I_n to be the matrix with 1's along the diagonal, and 0's everywhere else.

In particular, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Given a square $n \times n$ matrix A we define the *inverse* of A to be the $n \times n$ matrix A^{-1} such that $A \cdot A^{-1} = I_n$ where I_n is defined as in 4).

Fact 1: If A, B are $n \times n$ matrices then $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$; similarly, $(A \cdot B)^T = B^T \cdot A^T$.

1)* Prove the following:

a) $(I_n)^{-1} = I_n$

b) $(A^{-1})^{-1} = A$

c) $A^{-1} \cdot A = I_n$

d) $(A^T)^{-1} = (A^{-1})^T$

2) Calculate the inverse for the following matrices, or argue why it is impossible.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

3) Prove that if A is of the form $\begin{bmatrix} r & s \\ kr & ks \end{bmatrix}$ (k is a real number) then A does not have an inverse.

Given a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we define the *determinant of* A *as*

$$\det(A) = ad - bc.$$

4) Calculate the determinant of the following matrices:

a) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

d) $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

5) Prove the following facts about determinants ($A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$):

a) if $\det(A) = 0$ then A does not have an inverse. Hint: Problem 3!

b) if $\det(A) \neq 0$ then $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

c) $\det(A^T) = \det(A)$.

d) $\det(kA) = k^2 \det(A)$.

e) $\det(A^{-1}) = (\det(A))^{-1}$.

Fact 2: If A, B are matrices, then $\det(AB) = \det(A) \cdot \det(B)$.

Linear Independence

Let $\vec{x}_1, \ldots, \vec{x}_n$ be a collection of vectors. If another vector $\vec{y}$ can be written as $\vec{y} = c_1\vec{x_1} + \cdots + c_n\vec{x_n}$ (for real numbers $c_1, \ldots, c_n$) we say that $\vec{y}$ is a *linear combination* of $\vec{x_1}, \ldots, \vec{x_n}$.

6) Write $\vec{y}$ as a linear combination of $\vec{x}_1, \vec{x}_2$ or say why it is impossible for the following:

a) $\vec{y} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

b) $\vec{y} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

c) $\vec{y} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} -3 \\ -6 \end{bmatrix}$.

d) $\vec{y} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{x}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

A collection of vectors (i.e. $n \times 1$ matrices) $\vec{x}_1, \ldots, \vec{x}_n$ is *linearly dependent* if and only if there are real numbers $c_1, \ldots, c_n$ (at least one non-zero) such that $c_1\vec{x_1} + \cdots + c_n\vec{x_n} = \vec{0}$. Otherwise, the collection of vectors is called *linearly independent*.

From now on, we'll work with vectors of the form $\begin{bmatrix} a \\ b \end{bmatrix}$.

7) a) Give collections of size one, two, and three that are linearly dependent.

b) Give collections of size one and two that are linearly independent.

8) a) Prove that if $\vec{x_1}, \ldots, \vec{x_n}$ is linearly dependent, then one of $\vec{x_1}, \ldots, \vec{x_n}$ can be written as a linear combination of the remaining vectors.

b) Prove that $\vec{x} = \begin{bmatrix} a \\ b \end{bmatrix}, \vec{y} = \begin{bmatrix} c \\ d \end{bmatrix}$ are linearly dependent if and only if $\det\left(\begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}\right) = \det\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}\right) = 0$.

Fact 3: If $\vec{x}, \vec{y}, \vec{z}$ are 2×1 vectors, then $\vec{x}, \vec{y}, \vec{z}$ are linearly dependent.

9) Argue geometrically that Fact 3 is correct.

If $\vec{x}, \vec{y}$ are 2×1 vectors, we say $\vec{x}$ and $\vec{y}$ are a *basis for* $\mathbb{R}^2$ if any 2×1 vector $\vec{z}$ can be written as a linear combination of $\vec{x}$ and $\vec{y}$.

10)* Prove that $\vec{x}$ and $\vec{y}$ are a *basis* if and only if $\vec{x}$ and $\vec{y}$ are a linearly independent.

11) Are the following collections a basis for $\mathbb{R}^2$? If so, write $\begin{bmatrix} a \\ b \end{bmatrix}$ as a linear combination. Hint: Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination first.

a) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

b) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

c) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}$