INTRODUCTION TO MATRICES

MATH CIRCLE (HS1) 1/19/2014

Vectors

Recall the definition of a vector $\vec{x} = \langle x_1, \ldots, x_m \rangle = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$ from last week.

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$, and c is a real number, we defined addition of vectors and scalar multiplication as

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_m + y_m \end{bmatrix} \text{ and } c \cdot \vec{x} = \begin{bmatrix} cx_1 \\ \vdots \\ cx_m \end{bmatrix}$$

We also defined the dot product of two vectors as $\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots x_m y_m$.

Now define the *transpose* of a vector as $\vec{x}^T = \begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$. That is, we turn the one column of $\vec{x}$ into a row.

Matrices

Note we can view a vector as consisting of one column and some number m of rows. What happens if we want to have more than one row?

An $m \times n$ *matrix* has m rows and n columns:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

Therefore, a vector is just a matrix with one column (i.e. an $m \times 1$ matrix).

If will often be useful to view A as a collection of column vectors:

$$A = \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} \text{ where } \vec{c}_1 = \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix}, \cdots, \vec{c}_n = \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}.$$

Likewise we can view A as a collection of row vectors:

$$A = \begin{bmatrix} \vec{r}_1^T \\ \vdots \\ \vec{r}_m^T \end{bmatrix} \text{ where } \vec{r}_1^T = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \end{bmatrix}, \cdots, \vec{r}_n^T = \begin{bmatrix} a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

0) Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$ be an $m \times n$ matrix.

a) What are m, n?

b) Write A as a collection of column vectors. That is, write out $\vec{c}_1, \ldots, \vec{c}_n$.

c) Write A as a collection of row vectors. That is, write out $\vec{r}_1, \ldots, \vec{r}_n$.

Define addition of two $m \times n$ matrices componentwise, and scalar multiplication by a constant componentwise.

Note that this agrees with the definitions for vectors: if $A = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$, $B = \begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{bmatrix}$, and c is a real constant then

$$A + B = \begin{bmatrix} \vec{v}_1 + \vec{w}_1 & \cdots & \vec{v}_n + \vec{w}_n \end{bmatrix} \text{ and } cA = \begin{bmatrix} c\vec{v}_1 & \cdots & c\vec{v}_n \end{bmatrix}.$$

Further, define the transpose of A, $A^T = \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_n \end{bmatrix}$ (that is, swapping rows and columns).

Note that the transpose turns an $m \times n$ matrix into a $n \times m$ one.

1) Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & -1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & 4 \end{bmatrix}$. Calculate the following, or say why it doesn't make sense:

a) $A + B$

b) $C + B$

c) $3 \cdot B$

d) B^T

e) $C + A^T$

f) $A + 2B + 3C^T$

Let the $l \times m$ matrix $A = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_l^T \end{bmatrix}$ (row vectors) and the $m \times n$ matrix $B = \begin{bmatrix} \vec{w}_1 & \cdots & \vec{w}_n \end{bmatrix}$ (column vectors).

Define matrix multiplication $A \cdot B$ to be the $l \times n$ matrix

$$A \cdot B = \begin{bmatrix} \vec{v}_1 \cdot \vec{w}_1 & \vec{v}_1 \cdot \vec{w}_2 & \cdots & \vec{v}_1 \cdot \vec{w}_n \\ \vec{v}_2 \cdot \vec{w}_1 & \vec{v}_2 \cdot \vec{w}_2 & \cdots & \vec{v}_2 \cdot \vec{w}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{v}_l \cdot \vec{w}_1 & \vec{v}_l \cdot \vec{w}_2 & \cdots & \vec{v}_l \cdot \vec{w}_n \end{bmatrix}.$$

This works particularly well with *square matrices* as if A, B are both $n \times n$ matrices, $A \cdot B$ is another $n \times n$ square matrix.

2) Let A, B, C be as in 1), $D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 0 \\ 4 & -3 \end{bmatrix}$. Calculate the following, or say why it doesn't make sense.

a) $A \cdot B$

b) $A \cdot C$

c) $B \cdot C$

d) $C \cdot A$

e) $B \cdot A$

f) $C^T \cdot B^T$

g) $A^T \cdot B^T$

h) $D \cdot E$

i) $E \cdot D$

3) Let A, B be arbitrary $n \times n$ matrices.

a) Is it true that $A \cdot B = B \cdot A$?

b) Guess a formula for $A^T \cdot B^T$. Verify your guess works for D, E as in 2).

4) Suppose we are only dealing with square $n \times n$ matrices. We have seen that adding and multiplying these matrices gives us more $n \times n$ matrices.

a) How would you define a "zero matrix", denoted 0_n? What properties does this matrix share with the number 0?

b) How would you define a "one/identity matrix", denoted I_n? What properties does this matrix share with the number 1?

Given a square $n \times n$ matrix A we define the *inverse* of A to be the $n \times n$ matrix A^{-1} such that $A \cdot A^{-1} = I_n$ where I_n is defined as in 4).

Fact: If A, B are $n \times n$ matrices then $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

5)* Prove the following:

a) $(I_n)^{-1} = I_n$ and $(I_n)^T = I_n$

b) $(A^{-1})^{-1} = A$

c) $(A^T)^{-1} = (A^{-1})^T$

6) Calculate the inverse for the following matrices, or argue why it is impossible.

a) $\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

7)* a) Prove that if A is of the form $\begin{bmatrix} r & s \\ kr & ks \end{bmatrix}$ then A does not have an inverse.

b) Prove that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 0$ then A does not have an inverse.

Fact: The stronger claim holds: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then A has an inverse if and only if $ad - bc \neq 0$.