

# INTRODUCTION TO VECTORS

MATH CIRCLE (HS1) 1/12/2014

## Vectors in the Plane ( $\mathbb{R}^2$ )

One way to view a *vector* is as a line segment with a distinguished start and end point. Thus a vector is determined by its length and direction.

For example, when we can have a vector  $\vec{x}$  starting at the point  $(0, 0)$  and ending at  $(3, 4)$ . We write

$$\vec{x} = \langle 3, 4 \rangle \text{ or } \vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

We call 3 the *first component of  $\vec{x}$* ; similarly, 4 is the *second component of  $\vec{x}$* .

The length of  $\vec{x}$  is  $\sqrt{3^2 + 4^2} = 5$ , written  $\|\vec{x}\| = 5$ .

0) Determine the following vectors:

- The vector starting at  $(1, 1)$  and ending at  $(3, 4)$ .
- The vector starting at  $(-3, -2)$  and ending at  $(3, 4)$ .
- The vector starting at  $(a, b)$  and ending at  $(c, d)$ .
- The vector with the same direction as a) and twice as long.
- The vector with the opposite direction as b) and half as long.
- The vector with the same direction as  $\langle a, b \rangle$  except one unit long.

If we have two vectors  $\vec{x} = \langle a, b \rangle, \vec{y} = \langle c, d \rangle$  we define *addition* and *scalar multiplication*:

$$\vec{x} + \vec{y} = \langle a + c, b + d \rangle \text{ and if } c \text{ is a real number } c \cdot \vec{x} = \langle c \cdot a, c \cdot b \rangle.$$

(In this case we say we add and multiply *componentwise*.)

1) Let  $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{y} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{z} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ . First draw each of these vectors. Then calculate and draw each of:

- $\vec{x} + \vec{y}$
- $\vec{x} + \vec{z}$
- $\vec{x} - \vec{y}$
- $2 \cdot \vec{y}$
- $\vec{x} - 2\vec{y} + 3\vec{z}$

2) Explain addition and scalar multiplication geometrically.

3) Suppose  $\|\vec{x}\| = l$  and  $\|\vec{y}\| = m$ .

a) Calculate  $\|c \cdot \vec{x}\|$ .

b)\* What can you say about  $\|\vec{x} + \vec{y}\|$ ?

If we have two vectors  $\vec{x} = \langle a, b \rangle, \vec{y} = \langle c, d \rangle$  we define the *dot product* (or *scalar product*):

$$\vec{x} \cdot \vec{y} = a \cdot c + b \cdot d.$$

4) Let  $\vec{x}, \vec{y}, \vec{z}$  be as in Problem 1). Calculate each of  $\vec{x} \cdot \vec{y}, \vec{x} \cdot \vec{z}, \vec{y} \cdot \vec{z}$ .

5)\* a) Prove that  $(k\vec{x}) \cdot (l\vec{y}) = (k \cdot l) \cdot (\vec{x} \cdot \vec{y})$ .

b) Find  $\vec{x}, \vec{y}$  such that  $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\|$ .

c) Find  $\vec{x}, \vec{y}$  such that  $\vec{x} \cdot \vec{y} = -\|\vec{x}\| \|\vec{y}\|$ .

d) Find  $\vec{x}, \vec{y}$  such that  $\vec{x} \cdot \vec{y} = 0$ .

e) Try to explain what is happening in parts b)-d). Ask one of the teachers for the general formula!

### Vectors in Higher Dimensions ( $\mathbb{R}^n$ )

Note that everything we have done thus far can generalize to higher dimensions. Addition and scalar multiplication are still done componentwise, and if  $\vec{x} = \langle a_1, \dots, a_n \rangle, \vec{y} = \langle b_1, \dots, b_n \rangle$  then

$$\|\vec{x}\| = \sqrt{(a_1)^2 + \dots + (a_n)^2} \text{ and } \vec{x} \cdot \vec{y} = a_1 \cdot b_1 + \dots + a_n \cdot b_n.$$

As in the two dimensional case,  $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\| \cos(\theta)$ , where  $\theta$  is the angle between the two vectors. In particular:

- $\vec{x}$  and  $\vec{y}$  are perpendicular if and only if  $\vec{x} \cdot \vec{y} = 0$
- $\vec{x}$  and  $\vec{y}$  are parallel if and only if  $\vec{x} \cdot \vec{y} = \|\vec{x}\| \|\vec{y}\|$
- $\vec{x}$  and  $\vec{y}$  are antiparallel if and only if  $\vec{x} \cdot \vec{y} = -\|\vec{x}\| \|\vec{y}\|$

6) Classify the following vectors as perpendicular, parallel, or neither:

a)  $\langle 1, 2, 3, 4 \rangle$  and  $\langle 2, 3, 4, 5 \rangle$

b)  $\langle 2, 3, 4 \rangle$  and  $\langle 6, 9, 12 \rangle$

c)  $\langle 1, 0, 3, -2, 4 \rangle$  and  $\langle 2, -3, -2, -2, 0 \rangle$

d)  $\langle 1, 3, 5 \rangle$  and  $\langle -2, -6, -10 \rangle$

7) a) Come up with an easier way to tell if two vectors are parallel or antiparallel.

b)\* Prove your answer in a) algebraically in the two dimensional case.

**Linear Independence**

A collection of vectors  $\vec{x}_1, \dots, \vec{x}_n$  is *linearly dependent* if and only if there are real numbers  $c_1, \dots, c_n$  (at least one non-zero) such that  $c_1\vec{x}_1 + \dots + c_n\vec{x}_n = \vec{0}$ . Otherwise, the collection of vectors is called linearly independent.

Note: if  $\vec{y}$  can be written as  $\vec{y} = c_1\vec{x}_1 + \dots + c_n\vec{x}_n$  we say that  $\vec{y}$  is a *linear combination* of  $\vec{x}_1, \dots, \vec{x}_n$ .

8) Work in the plane (i.e.  $\mathbb{R}^2$ ) throughout this problem.

a) Give collections of size one, two, and three that are linearly dependent.

b) Give collections of size one and two that are linearly independent.

c) Argue geometrically that it is impossible to find three vectors that are linearly independent.

9) a) Prove that if  $\vec{x}_1, \dots, \vec{x}_n$  is linearly dependent, then one of  $\vec{x}_1, \dots, \vec{x}_n$  can be written as a linear combination of the remaining vectors.

b) Prove that  $\vec{x}, \vec{y}$  are linearly dependent if and only if they are parallel or antiparallel.

**Homework (For Further Investigation)**

1) Go back and review how we studied geometry using complex numbers last quarter. What similarities and differences do you see? Can you relate results we proved in problems 3) and 5) to things we proved about complex numbers?

2) Examine linear dependence and independence in three or more dimensions. How many linearly independent vectors can you find? Can you come up with a way to tell if vectors are linearly dependent or not geometrically? Etc.